Static Magnetic Field Computation – An Approach Based on the Domain-integrated Field Equations

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Abstract–**An unconventional algorithm is presented to compute quasi-static magnetic fields. It aims to be as close to the physics as possible for the class of strongly heterogeneous media. Using edge-element expansion functions for the magnetic field strength and face-element expansion functions for the magnetic flux density, a system of linear algebraic equations in the expansion coefficients is constructed from the exact satisfaction, for each element, of the domain-integrated field equations and of the compatibility relations, combined with the least-square satisfaction, for each element, of the constitutive relation. The resulting system of equations is over-determined and is solved by minimizing the L^2-norm of the residual. Accuracy and convergence are tested by applying the method to some two-dimensional problems whose solution is known analytically. A novel method to encompass an unbounded exterior domain is included in the method; it proves to perform remarkably well.**

Index terms–**Magnetostatics, nonhomogeneous media, numerical analysis**

I. Introduction

An unconventional algorithm to compute quasi-static magnetic fields in three-dimensional space is presented. It is based on the domain-integrated field equations. It can be implemented on a tetrahedral mesh, the faces and edges of which fit the boundaries of objects and interfaces between different media in the actual configuration up to order $O(h^2)$, where h is the mesh size parameter. The field variations in the interior of each tetrahedral cell are taken to be arbitrarily spatially linear, while the field expansion coefficients are chosen such that interface boundary conditions are satisfied within computational machine accuracy. Since the magnetic medium properties are assumed to vary continuously with position within each tetrahedron, the field representations are then correct up to and including order $O(h)$ as $h \to 0$ in each subdomain of the actual configuration not containing sharp edges or vertices (where the field becomes singular).

The entire configuration is embedded in a source-free vacuum environment. This enables the use of a source-type integral representation with the Green's function of the embedding to relate the values of the magnetic field on the boundary of the domain of computation to the ones in the interior domain of contrast in magnetic properties, which procedure mimics the presence of an unbounded exterior domain.

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II. Field equations

The quasi-static magnetic field in the configuration is characterized by the magnetic field strength $\boldsymbol{H}$ and the magnetic flux density $\boldsymbol{B}$, their support being $\mathbb{R}^3$. The excitation quantities are: the piecewise continuous volume density of electric current $\boldsymbol{J}$ (with support $\mathcal{D}^J$) and the piecewise continuous permanent magnetization $\boldsymbol{M}$ (with support $\mathcal{D}^M$). The magnetic properties of the material are characterized by the tensorial permeability $\boldsymbol{\mu}$ which varies piecewise continuously with position; the embedding has the permeability μ_0 $(= 4\pi \cdot 10^{-7}$ H/m). The support of $\boldsymbol{\mu} - \mu_0\boldsymbol{I}$, where $\boldsymbol{I}$ is the symmetric unit tensor of rank two, is $\mathcal{D}^\mu$. The equations to be solved are [1, Chapter 4]

$$\int_{\partial\mathcal{D}} \boldsymbol{n}\times\boldsymbol{H}\,\mathrm{d}A = \int_{\mathcal{D}} \boldsymbol{J}\,\mathrm{d}V, \tag{1}$$

$$\int_{\partial\mathcal{D}} \boldsymbol{n}\cdot\boldsymbol{B}\,\mathrm{d}A = 0, \tag{2}$$

for any bounded domain $\mathcal{D}$ and its boundary surface $\partial\mathcal{D}$ with outward unit normal $\boldsymbol{n}$, and the local constitutive equation

$$\boldsymbol{B} = \boldsymbol{\mu}\cdot\boldsymbol{H} + \mu_0\boldsymbol{M} \tag{3}$$

that is to hold in the open subdomain of the configuration where $\boldsymbol{\mu}$ and $\boldsymbol{M}$ vary continuously with position. Furthermore, (1) entails the the *compatibility relation*

$$\int_{\mathcal{S}} \boldsymbol{n}\cdot\boldsymbol{J}\,\mathrm{d}A = 0 \ \text{ for any closed surface } \mathcal{S}, \tag{4}$$

while (1), (2) and (4) entail the boundary conditions

$$\boldsymbol{n}\cdot\boldsymbol{J} = 0 \ \text{ on boundaries of coils}, \tag{5}$$

$$\boldsymbol{n}\times\boldsymbol{H} = \text{continuous across interfaces}, \tag{6}$$

$$\boldsymbol{n}\cdot\boldsymbol{B} = \text{continuous across interfaces}, \tag{7}$$

where $\boldsymbol{n}$ is the unit vector along the normal to the coil boundaries, or the interfaces of jump discontinuity in $\boldsymbol{\mu}$, respectively. Finally, $\boldsymbol{H}$ admits the source-type integral representation

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{r}) = &\boldsymbol{\nabla}\left\{\boldsymbol{\nabla}\cdot\left[\int_{\mathcal{D}^\mu}\frac{\boldsymbol{M}^{\rm ind}(\boldsymbol{r}')}{4\pi R}\,{\rm d}V(\boldsymbol{r}')\right]\right\}\\ &+\boldsymbol{\nabla}\left\{\boldsymbol{\nabla}\cdot\left[\int_{\mathcal{D}^M}\frac{\boldsymbol{M}(\boldsymbol{r}')}{4\pi R}\,{\rm d}V(\boldsymbol{r}')\right]\right\}\\ &+\boldsymbol{\nabla}\times\left[\int_{\mathcal{D}^J}\frac{\boldsymbol{J}(\boldsymbol{r}')}{4\pi R}\,{\rm d}V(\boldsymbol{r}')\right] \quad \text{for } \boldsymbol{r}\in\mathbb{R}^3\,, \quad (8)\end{aligned}$$

in which $R = |\boldsymbol{r}-\boldsymbol{r}'|$ and $\boldsymbol{M}^{\rm ind} = (\boldsymbol{\mu}_{\rm r}-\boldsymbol{I})\cdot\boldsymbol{H}$ is the induced contrast magnetization with respect to the embedding, with $\boldsymbol{\mu}_{\rm r} = \mu_0{}^{-1}\boldsymbol{\mu}$. From (8) it follows that

$$\boldsymbol{H}(\boldsymbol{r}) = O(|\boldsymbol{r}|^{-3}) \text{ as } |\boldsymbol{r}|\to\infty \text{ uniformly in all directions,} \quad (9)$$

where use is made of the Landau order symbol O [2, page 1019]. It is noted that (4) is a necessary condition for the existence of any solution at all. The problem thus formulated has a unique solution. This is also the case when on the boundary of the domain of computation either $\boldsymbol{n}\times\boldsymbol{H}$ or $\boldsymbol{n}\cdot\boldsymbol{B}$ have prescribed values ($\boldsymbol{n}$ = unit normal to the boundary). The latter values are often used in feasibility studies where test problems are analyzed (see Section IV).

III. Discretized field problem

For the discretization of the field problem a bounded domain of computation $\mathcal{D}^{\rm comp}$ is chosen such that its boundary surface $\partial\mathcal{D}^{\rm comp}$ is entirely located in the embedding. Next, $\mathcal{D}^{\rm comp}$ is subdivided into simplices (tetrahedra) that all have vertices, faces and edges in common and fit surfaces of discontinuity in excitation quantities or constitutive parameters up to $O(h^2)$. Each tetrahedral cell $\mathcal{T}$ has nodes at $\{\boldsymbol{r}(I);\ I=0,1,2,3\}$, outwardly oriented opposite vectorial faces $\{\boldsymbol{A}(I);\ I=0,1,2,3\}$ and vectorial edges $\{\boldsymbol{r}(J)-\boldsymbol{r}(I);\ I=0,1,2,3;\ J=0,1,2,3;\ J\neq I\}$. In it, the magnetic field strength's consistently linear edge-element representation is

$$\boldsymbol{H}(\boldsymbol{r}) = \sum_{I=0}^{3}\sum_{J=0|J\neq I}^{3}\alpha^{\boldsymbol{H}}(I,J)\,\boldsymbol{h}(I,J,\boldsymbol{r}), \quad (10)$$

where the expansion coefficients

$$\alpha^{\boldsymbol{H}}(I,J) = \boldsymbol{H}(\boldsymbol{r}(I))\cdot\frac{\boldsymbol{r}(J)-\boldsymbol{r}(I)}{|\boldsymbol{r}(J)-\boldsymbol{r}(I)|} \quad (11)$$

are the projections of $\boldsymbol{H}$ on the edges, and the (face-oriented) expansion functions read

$$\boldsymbol{h}(I,J,\boldsymbol{r}) = -\frac{|\boldsymbol{r}(J)-\boldsymbol{r}(I)|}{3V}\,\Phi(I,\boldsymbol{r})\,\boldsymbol{A}(J), \quad (12)$$

with

$$\Phi(I,\boldsymbol{r}) = \frac{1}{3V}\left\{1/4-\left[\boldsymbol{r}-\sum_{K=0}^{3}\boldsymbol{r}(K)\right]\cdot\boldsymbol{A}(I)\right\} \quad (13)$$

being the local, scalar, linear interpolation function. Furthermore, the magnetic flux density's consistently linear face-element representation is

$$\mu_0{}^{-1}\boldsymbol{B}(\boldsymbol{r}) = \sum_{I=0}^{3}\sum_{J=0|J\neq I}^{3}\alpha^{\boldsymbol{B}}(I,J)\,\boldsymbol{b}(I,J,\boldsymbol{r}), \quad (14)$$

where the expansion coefficients

$$\alpha^{\boldsymbol{B}}(I,J) = \mu_0{}^{-1}\boldsymbol{B}(\boldsymbol{r}(I))\cdot\frac{\boldsymbol{A}(J)}{|\boldsymbol{A}(J)|} \quad (15)$$

are the projections of $\boldsymbol{B}$ on the faces, and the (edge-oriented) expansion functions read

$$\boldsymbol{b}(I,J,\boldsymbol{r}) = -\frac{|\boldsymbol{A}(J)|}{3V}\,\Phi(I,\boldsymbol{r})\,[\boldsymbol{r}(J)-\boldsymbol{r}(I)]. \quad (16)$$

The scaling that we have applied in (10)–(15) makes all expansion coefficients to be expressed in the SI-unit [A/m] and is advantageous from a computational point of view. These equations (or similar ones that use the projections of $\boldsymbol{H}$ on the edges of a complex mesh and the projections of $\boldsymbol{B}$ on the latter's faces) are substituted in Eqs. (1) and (2) when applied to each tetrahedral cell. Next, the expansions are substituted in

$$\int_{\mathcal{T}}\left|\boldsymbol{B}-\boldsymbol{\mu}\cdot\boldsymbol{H}-\mu_0\boldsymbol{M}\right|^2{\rm d}V = \text{minimum}. \quad (17)$$

Using a standard least-squares minimization technique, Eq. (17) leads to the conditions

$$\begin{aligned}\sum_{I=0}^{3}\sum_{J=0|J\neq I}^{3}&\left\{\alpha^{\boldsymbol{B}}(I,J)\int_{\mathcal{T}}\boldsymbol{h}(K,L,\boldsymbol{r})\cdot\boldsymbol{b}(I,J,\boldsymbol{r})\,{\rm d}V\right.\\ &\left.-\alpha^{\boldsymbol{H}}(I,J)\int_{\mathcal{T}}\boldsymbol{h}(K,L,\boldsymbol{r})\cdot[\boldsymbol{\mu}_{\rm r}\cdot\boldsymbol{h}(I,J,\boldsymbol{r})]\,{\rm d}V\right\}\\ &=\int_{\mathcal{T}}\boldsymbol{h}(K,L,\boldsymbol{r})\cdot\boldsymbol{M}\,{\rm d}V, \quad (18)\end{aligned}$$

$$\begin{aligned}\sum_{I=0}^{3}\sum_{J=0|J\neq I}^{3}&\left\{\alpha^{\boldsymbol{B}}(I,J)\int_{\mathcal{T}}\boldsymbol{b}(K,L,\boldsymbol{r})\cdot\boldsymbol{b}(I,J,\boldsymbol{r})\,{\rm d}V\right.\\ &\left.-\alpha^{\boldsymbol{H}}(I,J)\int_{\mathcal{T}}\boldsymbol{b}(K,L,\boldsymbol{r})\cdot[\boldsymbol{\mu}_{\rm r}\cdot\boldsymbol{h}(I,J,\boldsymbol{r})]\,{\rm d}V\right\}\\ &=\int_{\mathcal{T}}\boldsymbol{b}(K,L,\boldsymbol{r})\cdot\boldsymbol{M}\,{\rm d}V, \quad (19)\end{aligned}$$

for $K=0,1,2,3$ and $L=0,1,2,3$ with $L\neq K$. Finally, the values of $\boldsymbol{n}\times\boldsymbol{H}$ on $\partial\mathcal{D}^{\rm comp}$ (which are needed to formulate a uniquely solvable problem) are interrelated to the values of $\boldsymbol{M}^{\rm ind}$, $\boldsymbol{M}$ and $\boldsymbol{J}$ in $\mathcal{D}^\mu$, $\mathcal{D}^M$ and $\mathcal{D}^J$, respectively, via (8). From the local relations that follow from (10)–(19), the global sequence of expansion coefficients and its corresponding system's matrix are constructed by invoking the continuity conditions (6) and (7).

The total procedure leads to an over-determined system of linear, algebraic equations that is subsequently solved by minimizing the L^2-norm of the residual. The latter

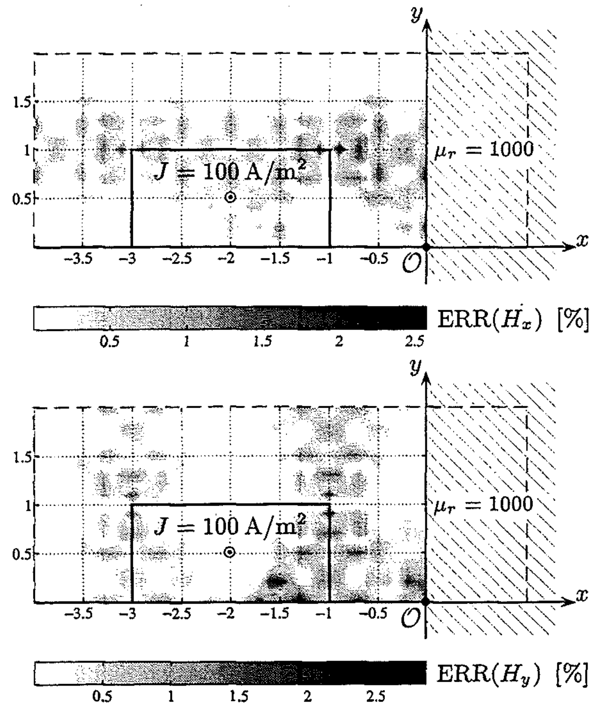


Fig. 1. Local relative error in computed magnetic field strength for $x \in [-4, 0]$ and $y \in [0, 2]$; impressed boundary conditions: tangential components of magnetic field strength; mesh size parameter $h = 0.25$ m; explicit boundary conditions according to the exact solution are imposed on the boundary of the domain of computation.

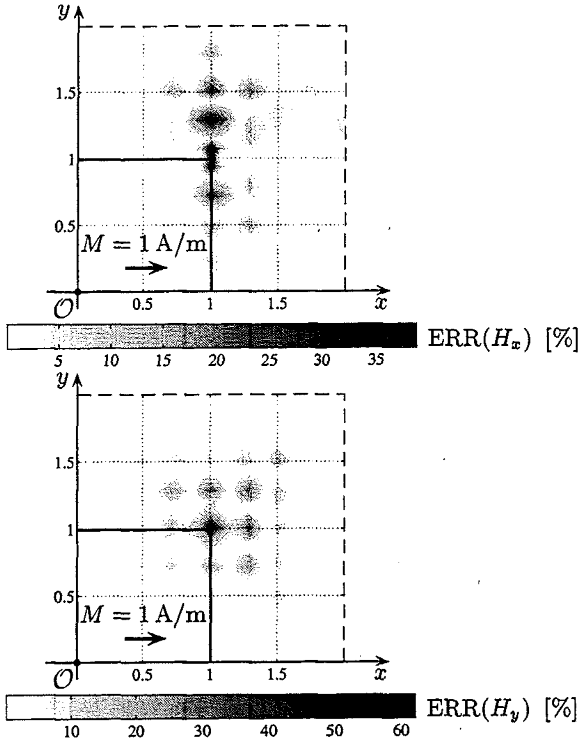


Fig. 2. Local relative error in computed magnetic field strength for $x \in [0, 2]$ and $y \in [0, 2]$; impressed boundary conditions: tangential components of magnetic field strength for $\{x = 0\}$, $\{y = 2\}$ and $\{x = 2\}$ and normal components of magnetic flux density for $\{y = 0\}$; mesh size parameter $h = 0.25$ m; explicit boundary conditions according to the exact solution are imposed on the boundary of the domain of computation.

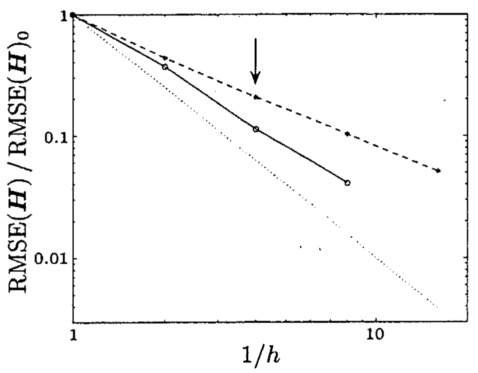
procedure requires the evaluation of the product of the matrix of global expansion coefficients and its transpose. The relevant product is evaluated efficiently by keeping track of the band structure of the matrix and carrying out the operation of matrix multiplication on the (much smaller) submatrices of this bandwidth, which essentially amounts to summing the relevant element contributions. (In this manner, our method can be easily implemented in a general-purpose finite-element environment.) This feature of efficiency is expected to substantially counterbalance the increased computational effort accompanying the use of the larger number of coefficients associated with the expansions of both $\boldsymbol{H}$ and $\boldsymbol{B}$.

Other "dual" approaches, in the realm of finite-element formulations, have been presented in [3]–[7]. Here, [3]–[6] deal with full Maxwell fields. Second-order differential equations are used for the field vectors, upon which the flux densities are obtained by differentiation in accordance with Maxwell's equations. Subsequently, the constitutive relations are invoked via the minimization of some positive definite functional that differs from (17). In this approach, the spatial differentiations invariably lead to spurious "surface charges" or "surface currents" that are absent in the real physical problem. In [7], weighted forms of the magnetostatic curl and divergence differential equations, with edge expansion for the field and face expansion for the flux density are used, upon which the constitutive relation as in (17) is invoked. Here, it can be argued that our Eqs. (1) and (2) more closely model the physics (in strongly heterogeneous media) than some weighted form of curl and divergence differential equations.

IV. Test problems

To investigate the feasibility, accuracy, and convergence properties of the method, four test problems whose solutions are known analytically are taken. They incorporate all physical features that show up in a quasi-static magnetic field: excitation by an impressed volume current, excitation by an impressed magnetization, and the presence of a high-contrast jump discontinuity in the permeability of the media. The four problems are:

1. the two-dimensional magnetic field generated by an impressed electric volume current uniformly distributed over a conductor with rectangular cross-section, in free space,
2. the two-dimensional magnetic field generated by an impressed electric volume current uniformly distributed over a conductor with rectangular cross-section, in the presence of a semi-infinite medium with constant permeability,
3. the two-dimensional magnetic field generated by an impressed magnetization uniformly distributed over a rectangular domain in free space,

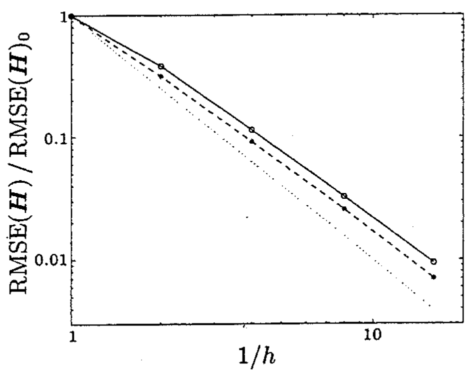


⋯⋯ Quadratic convergence

—— Rectangular conductor in a two-media configuration
$\mathrm{RMSE}(\boldsymbol{H})_0 = 3.65\,\%$

– – – Uniformly magnetized rectangular domain in free space
$\mathrm{RMSE}(\boldsymbol{H})_0 = 14.13\,\%$

Fig. 3. Convergence of the numerical solution with mesh refinement in the case when explicit boundary conditions according to the exact solution are imposed on the boundary of the domain of computation; the arrows indicate the mesh size used in Figs. 1 and 2.



⋯⋯ Quadratic convergence

—— Using explicit boundary conditions
$\mathrm{RMSE}(\boldsymbol{H})_0 = 3.89\,\%$

– – – Using the contrast source formulation
$\mathrm{RMSE}(\boldsymbol{H})_0 = 4.89\,\%$

Fig. 4. Convergence of the numerical solution with mesh refinement for a rectangular conductor in free space in the case when the boundary conditions using the contrast source formulation are imposed.

4. the two-dimensional magnetic field generated by an impressed magnetization uniformly distributed over a rectangular domain, in the presence of a semi-infinite medium with constant permeability.

A square, uniform mesh is chosen with the local scalar expansion functions being the Cartesian product of the ones interpolating linearly the field values along the edges. The program is written in MATLAB™.

In some cases, the values of the field components on the outer boundary of the domain of computation needed to formulate a uniquely solvable problem are taken to be the exact ones, in other cases we have computed these values numerically from Eq. (8). Local relative errors are shown in Figs. 1 and 2, where the relevant configuration is indicated in the figure. The rate of convergence is shown in Figs. 3 and 4.

For the problems with impressed electric volume current the decrease in error is quadratic with decreasing mesh size (as expected in view of the piecewise linear representations of the field quantities). For the problem with impressed magnetization the decrease in error is linear with decreasing mesh size; this slower rate of convergence can be ascribed to the fact that the exact solution shows a logarithmic singularity at the rim of the magnetic pole planes of the domain of impressed magnetization.

Finally, the use of (8) has been tested by evaluating from it numerically the values of $\boldsymbol{n}\times\boldsymbol{H}$ on $\partial\mathcal{D}^{\mathrm{comp}}$. Even the use of the crudest (rectangle) integration rule proved to give rise to an increase of $\mathrm{RMSE}(\boldsymbol{H}_0)$ of one percent only (Fig. 4).

V. Discussion of the results

The analysis has indicated that the domain-integration method could be an attractive alternative to both the finite-difference and finite-element methods in field computation. It is felt that, especially in strongly inhomogeneous media, where the field values are not differentiable over any domain of appreciable size, our method comes closest to the physics of the problem. Further studies, applied to realistic three-dimensional configurations of sufficient complexity and on a tetrahedral mesh, are needed to assess the detailed features of the method. A first step towards downsizing the system of equations in the expansion coefficients is to use nodal elements rather than edge or face elements at all locations where the field quantities vary continuously and reserving the edge- and face-element expansions for those locations where the medium properties do show discontinuities.

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