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TIME-DOMAIN RECIPROCITY THEOREMS FOR ELASTODYNAMIC WAVE FIELDS IN SOLIDS WITH RELAXATION AND THEIR APPLICATION TO INVERSE PROBLEMS

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Time-domain reciprocity theorems of the time-convolution and the time-correlation type for elastodynamic wave fields in linear, time-invariant, and locally reacting solids are discussed. Inhomogeneity, anisotropy, and arbitrary relaxation effects, both of the active (anti-causal) and passive (causal) kind, are included. The analysis is entirely carried out in space-time, without intermediate recourse to the frequency or the wavevector domains. The application to inverse source and inverse constituency (inverse profiling or scattering) problems is discussed.

1. Introduction

A wave field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two admissible physical states that could occur in one and the same domain in space-time. As far as elastodynamic wave fields are concerned, Betti [1] is commonly credited to be the first to derive a reciprocity theorem relating to elasticity; it applies to static elastic fields. Lord Rayleigh was apparently the first to give an elastodynamic reciprocity theorem [2]; it applies to time-harmonic vibrations in a mechanical system. For arbitrary time variations of the wave motion in a linear, time-invariant medium, Bojarski [3] clearly distinguished between convolution- and correlation-type reciprocity relations and he presented the corresponding time-domain reciprocity theorems for homogeneous, isotropic, and lossless media, both for acoustic wave fields in fluids and for electromagnetic waves. In this connection, he introduced the concept of "effectual" (or "effectal") wave field as the time-reversed counterpart of a "causal" wave field and emphasized the relationship between time-advanced and time-retarded wave fields. A correlation-type elastodynamic reciprocity theorem (in the present terminology) for correlation duration zero was given by Lamb [4] (see also Love [5] and Teodorescu [6]). The convolution-type reciprocity relation for elastodynamic wave fields in homogeneous, isotropic, perfectly elastic solids was derived by Graffi [7-10] (see also Wheeler and Sternberg [11], Achenbach [12], and Eringen and Suhubi [13]). Payton [14] applied these to moving point-load problems. De Hoop [15] generalized the time-convolution reciprocity theorem to inhomogeneous, anisotropic, viscoelastic solids. The application of reciprocity theorems to inverse scattering is reviewed by Fisher and Langenberg [16], where an extensive list of references to earlier papers on this subject can be found.

The present investigation deals with time-convolution and time-correlation reciprocity theorems for elastodynamic wave fields in time-invariant configurations that are linear and locally reacting in their elastodynamic behavior. As regards the space-time geometry in which the two admissible states occur, this implies that this geometry is the Cartesian product $D \times \mathbb{R}$ of a time-invariant spatial domain $D \subset \mathbb{R}^3$

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and the real time axis $\mathbb{R}$. Further, the constitutive parameters of the media present in the two states are time invariant and independent of the wave field values. No further restrictions are imposed. Inhomogeneity and arbitrary anisotropy are included, as well as arbitrary relaxation effects. Both the time-convolution and the time-correlation type of reciprocity theorem have an important field of application in inverse source and inverse constituency (inverse profiling or scattering) problems. These applications will be discussed from a general point of view.

The position of observation in $\mathbb{R}^3$ is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin O and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. In the indicated order the base vectors form a right-handed system. The subscript notation for Cartesian vectors and tensors in $\mathbb{R}^3$ is employed and the summation convention applies. The corresponding lower case Latin subscripts are to be assigned the values $\{1, 2, 3\}$. Whenever appropriate, the position vector will be denoted by $\boldsymbol{x} = x_p \boldsymbol{i}_p$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_p denotes the differentiation with respect to x_p, ∂_t denotes the differentiation with respect to t.

The reciprocity theorems will be derived for a bounded domain D. In the analysis also the boundary ∂D of D occurs. The unit vector along the normal to ∂D is denoted by ν_m; it points away from D (Fig. 1).

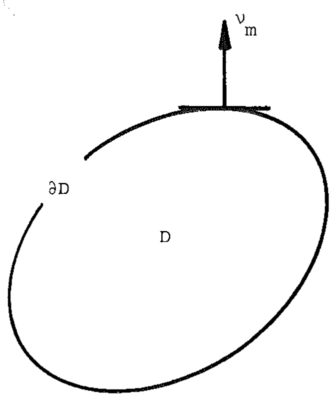

Fig. 1. Time-invariant configuration to which the reciprocity theorems apply.

2. Some properties of the time convolution and the time correlation of space-time functions

In this section we present the properties of the time convolution and the time correlation of space-time functions as far as they are needed in the derivation of the reciprocity theorems. Let $f_1 = f_1(\boldsymbol{x}, t)$ and $f_2 = f_2(\boldsymbol{x}, t)$ be two transient space-time functions. By this we mean that the functions are absolutely integrable on the entire $t \in \mathbb{R}$. Then, the time convolution of f_1 and f_2 is defined as

$$C(f_1, f_2; \boldsymbol{x}, \tau) = \int_{t \in \mathbb{R}} f_1(\boldsymbol{x}, t) f_2(\boldsymbol{x}, \tau - t)\, \mathrm{d}t$$

$$= \int_{t \in \mathbb{R}} f_1(\boldsymbol{x}, \tau - t) f_2(\boldsymbol{x}, t)\, \mathrm{d}t = C(f_2, f_1; \boldsymbol{x}, \tau). \tag{2.1}$$

Equation (2.1) shows that the time convolution is symmetrical in f_1 and f_2. The time correlation of f_1 and f_2 is defined as

$$R(f_1, f_2; \boldsymbol{x}, \tau) = \int_{t\in\mathbb{R}} f_1(\boldsymbol{x}, t) f_2(\boldsymbol{x}, t-\tau)\,\mathrm{d}t$$
$$= \int_{t\in\mathbb{R}} f_1(\boldsymbol{x}, t+\tau) f_2(\boldsymbol{x}, t)\,\mathrm{d}t = R(f_2, f_1; \boldsymbol{x}, -\tau). \tag{2.2}$$

Equation (2.2) shows that the time correlation is not symmetrical in f_1 and f_2.

Let, now, $\bar{f}$ denote the time-reversed of f, i.e.,

$$\bar{f}(\boldsymbol{x}, t) = f(\boldsymbol{x}, -t). \tag{2.3}$$

Then, it follows from (2.1)-(2.3) that

$$R(f_1, f_2; \boldsymbol{x}, \tau) = C(f_1, \bar{f}_2; \boldsymbol{x}, \tau). \tag{2.4}$$

Using (2.1), we further obtain the properties

$$C(\bar{f}_1, f_2; \boldsymbol{x}, \tau) = \bar{C}(f_1, \bar{f}_2; \boldsymbol{x}, \tau) \tag{2.5}$$

and

$$C(\bar{f}_1, \bar{f}_2; \boldsymbol{x}, \tau) = \bar{C}(f_1, f_2; \boldsymbol{x}, \tau). \tag{2.6}$$

For the time derivative of the time convolution the rules

$$\partial_\tau C(f_1, f_2; \boldsymbol{x}, \tau) = C(f_1, \partial_t f_2; \boldsymbol{x}, \tau) = C(\partial_t f_1, f_2; \boldsymbol{x}, \tau) \tag{2.7}$$

apply. In view of the property

$$\overline{\partial_t f} = -\partial_t \bar{f}, \tag{2.8}$$

the time derivatives of the time correlation are taken care of by using (2.4) in conjunction with (2.7), i.e.,

$$\partial_\tau C(f_1, \bar{f}_2; \boldsymbol{x}, \tau) = C(\partial_t f_1, \bar{f}_2; \boldsymbol{x}, \tau) = -C(f_1, \overline{\partial_t f_2}; \boldsymbol{x}, \tau). \tag{2.9}$$

For the incorporation of relaxation effects in the reciprocity theorems we also need the time convolution of three space-time functions. For this, either of the definitions

$$C(f_1, f_2, f_3; \boldsymbol{x}, \tau) = C(f_1, C(f_2, f_3); \boldsymbol{x}, \tau) = C(C(f_1, f_2), f_3; \boldsymbol{x}, \tau) \tag{2.10}$$

holds.

In view of its simpler properties, the time-convolution concept is used throughout the entire subsequent derivations, i.e., both for the time-convolution and for the time-correlation reciprocity theorems.

3. Properties of the elastodynamic wave field in the configuration

In each subdomain of the configuration where the elastodynamic properties vary continuously with position, the elastodynamic wave field quantities are continuously differentiable and satisfy the equations

$$-\Delta_{k,m,p,q}\partial_m \tau_{p,q} + \dot{\Phi}_k = f_k, \tag{3.1}$$

$$\Delta_{i,j,m,r}\partial_m v_r - \dot{e}_{i,j} = h_{i,j}, \tag{3.2}$$

where

- $\tau_{p,q}$ = stress (Pa),
- v_r = particle velocity ($\mathrm{m \cdot s^{-1}}$),
- $\dot{\Phi}_k$ = mass flow density rate ($\mathrm{kg \cdot m^{-2} \cdot s^{-2}}$),
- $\dot{e}_{i,j}$ = deformation rate ($\mathrm{s^{-1}}$),
- f_k = volume source density of force ($\mathrm{N \cdot m^{-3}}$),
- $h_{i,j}$ = volume source density of strain rate ($\mathrm{s^{-1}}$),
- $\Delta_{i,j,m,r} = \frac{1}{2}(\delta_{i,m}\delta_{j,r} + \delta_{i,r}\delta_{j,m})$,

and $\delta_{i,m}$ is the symmetrical unit tensor of rank two (Kronecker tensor). Equations (3.1) and (3.2) are supplemented by the constitutive relations. For a linear, time-invariant, locally reacting solid these are

$$\dot{e}_{i,j} = \partial_t \int_{\tau=-\infty}^{\infty} \kappa_{i,j,p,q}(\boldsymbol{x}, \tau)\tau_{p,q}(\boldsymbol{x}, t-\tau)\, \mathrm{d}\tau, \tag{3.3}$$

$$\dot{\Phi}_k = \partial_t \int_{\tau=-\infty}^{\infty} \gamma_{k,r}(\boldsymbol{x}, \tau)v_r(\boldsymbol{x}, t-\tau)\, \mathrm{d}\tau, \tag{3.4}$$

where

- $\kappa_{i,j,p,q}$ = compliance relaxation function ($\mathrm{Pa^{-1} \cdot s^{-1}}$),
- $\gamma_{k,r}$ = inertia relaxation function ($\mathrm{kg \cdot m^{-3} \cdot s^{-1}}$).

Using the notation of (2.1), (3.3) and (3.4) can be rewritten as

$$\dot{e}_{i,j}(\boldsymbol{x}, t) = \partial_t C(\kappa_{i,j,p,q}, \tau_{p,q}; \boldsymbol{x}, t), \tag{3.5}$$

$$\dot{\Phi}_k(x, t) = \partial_t C(\gamma_{k,r}, v_r; \boldsymbol{x}, t), \tag{3.6}$$

respectively. In (3.3) and (3.4), inhomogeneity, anisotropy and relaxation of the solid are included. It is noted that the type of anisotropy considered here has been introduced by Bromwich [17] (see also Love [18]); relaxation effects of the indicated kind have been introduced by Boltzmann [19] and, in a more general setting, by Volterra [20, 21] (see also Chao and Achenbach [22] and Gurtin and Sternberg [23]).

If $\{\kappa_{i,j,p,q}, \gamma_{k,r}\}(\boldsymbol{x}, \tau) = 0$ when $\tau < 0$, the solid at $\boldsymbol{x}$ is causal, to use the terminology of linear, time-invariant systems. If

$$\kappa_{i,j,p,q}(\boldsymbol{x}, \tau) = s_{i,j,p,q}(\boldsymbol{x})\delta(\tau), \tag{3.7}$$

$$\gamma_{k,r}(\boldsymbol{x}, \tau) = \rho_{k,r}(\boldsymbol{x})\delta(\tau), \tag{3.8}$$

where $\delta(\tau)$ is the unit impulse (Dirac distribution), the solid is instantaneously reacting, and $s_{i,j,p,q}$ and $\rho_{k,r}$ are its compliance and its (tensorial) volume density of mass, respectively. If $\{\kappa_{i,j,p,q}, \gamma_{k,r}\}(\boldsymbol{x}, \tau) = 0$ when $\tau > 0$, the solid is anticausal or effectual. From an energetic point of view, a solid for which $\{\kappa_{i,j,p,q}, \gamma_{k,r}\}(\boldsymbol{x}, \tau) \neq 0$ when $\tau > 0$ is dissipative, a solid for which (3.7) and (3.8) hold is lossless, and a solid for which $\{\kappa_{i,j,p,q}, \gamma_{k,r}\}(\boldsymbol{x}, \tau) \neq 0$ when $\tau < 0$ is active. A solid that is either dissipative or lossless is also denoted as passive. For our reciprocity theorems no specific type of relaxation function is presupposed. Conservation of energy of course requires an elastodynamically active solid to be stimulated, through the constitutive parameters, by some other physical phenomenon (for example, by the passage of light through it).

It is assumed that $\kappa_{i,j,p,q}$ and $\gamma_{k,r}$ are piecewise continuous functions of position. At an interface between two different solids, at which we assume the solids to be in rigid contact, the constitutive parameters jump by finite amounts. Across such an interface the traction (i.e., the normal component of the stress) and the particle velocity are continuous. If an elastically impenetrable object is present, either the traction (at a void) or the particle velocity (at an immovable rigid object) has zero value at its boundary. Through the pertaining boundary conditions the presence of either interfaces or impenetrable objects is accounted for in the reciprocity relations.

The two states that occur in the reciprocity theorems will be denoted by the superscripts "a" and "b", respectively. It is noted that the two states can apply to different source distributions and to solids with different properties, but they must be present in one and the same domain in space-time.

4. The reciprocity theorem of the time-convolution type

The reciprocity theorem of the time-convolution type follows upon considering the interaction quantity

$$\Delta_{m,r,p,q}[C(-\tau^{\mathrm{a}}_{p,q}, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau)].$$

Using (3.1) and (3.2) for each of the two states we obtain

$$\Delta_{m,r,p,q}\partial_m C(-\tau^{\mathrm{a}}_{p,q}, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) = C(f^{\mathrm{a}}_r - \dot{\Phi}^{\mathrm{a}}_r, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) + C(-\tau^{\mathrm{a}}_{p,q}, h^{\mathrm{b}}_{p,q} + \dot{e}^{\mathrm{b}}_{p,q}; \boldsymbol{x}, \tau) \tag{4.1}$$

and

$$\Delta_{m,r,p,q}\partial_m C(-\tau^{\mathrm{b}}_{p,q}, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau) = C(f^{\mathrm{b}}_r - \dot{\Phi}^{\mathrm{b}}_r, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau) + C(-\tau^{\mathrm{b}}_{p,q}, h^{\mathrm{a}}_{p,q} + \dot{e}^{\mathrm{a}}_{p,q}; \boldsymbol{x}, \tau). \tag{4.2}$$

Now, in view of (3.5) and (3.6) we have

$$C(\dot{\Phi}^{\mathrm{b}}_r, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau) - C(\dot{\Phi}^{\mathrm{a}}_r, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) = \partial_\tau C(\gamma^{\mathrm{b}}_{r,k} - \gamma^{\mathrm{a}}_{k,r}, v^{\mathrm{a}}_r, v^{\mathrm{b}}_k; \boldsymbol{x}, \tau) \tag{4.3}$$

and

$$C(-\tau^{\mathrm{a}}_{p,q}, \dot{e}^{\mathrm{b}}_{p,q}; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, \dot{e}^{\mathrm{a}}_{p,q}; \boldsymbol{x}, \tau) = -\partial_\tau C(\kappa^{\mathrm{b}}_{p,q,i,j} - \kappa^{\mathrm{a}}_{i,j,p,q}, \tau^{\mathrm{a}}_{p,q}, \tau^{\mathrm{b}}_{i,j}; \boldsymbol{x}, \tau), \tag{4.4}$$

where (2.7) has been used. Subtracting (4.2) from (4.1) and employing (4.3) and (4.4), we arrive at

$$\begin{aligned}
&\Delta_{m,r,p,q}\partial_m[C(-\tau^{\mathrm{a}}_{p,q}, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau)]\\
&\quad = \partial_\tau C(\gamma^{\mathrm{b}}_{r,k} - \gamma^{\mathrm{a}}_{k,r}, v^{\mathrm{a}}_r, v^{\mathrm{b}}_k; \boldsymbol{x}, \tau) - \partial_\tau C(\kappa^{\mathrm{b}}_{p,q,i,j} - \kappa^{\mathrm{a}}_{i,j,p,q}, \tau^{\mathrm{a}}_{p,q}, \tau^{\mathrm{b}}_{i,j}; \boldsymbol{x}, \tau)\\
&\qquad + C(f^{\mathrm{a}}_r, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) + C(-\tau^{\mathrm{a}}_{p,q}, h^{\mathrm{b}}_{p,q}; \boldsymbol{x}, \tau) - C(f^{\mathrm{b}}_r, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, h^{\mathrm{a}}_{p,q}; \boldsymbol{x}, \tau).
\end{aligned} \tag{4.5}$$

Equation (4.5) is the local form of the time-convolution reciprocity theorem. The first two terms in the right-hand side are representative for the differences in the properties of the solids present in the two states; they vanish at those locations where $\gamma^{\mathrm{b}}_{r,k}(\boldsymbol{x}, \tau) = \gamma^{\mathrm{a}}_{k,r}(\boldsymbol{x}, \tau)$ and $\kappa^{\mathrm{b}}_{p,q,i,j}(\boldsymbol{x}, \tau) = \kappa^{\mathrm{a}}_{i,j,p,q}(\boldsymbol{x}, \tau)$ for all $\tau \in \mathbb{R}$. In case the latter conditions hold, the two media are denoted as each other's adjoints. Note in this respect that the adjoint of a causal (effectual) medium is causal (effectual), too. The last four terms in the right-hand side of (4.5) are associated with the source distributions; they vanish at those locations where no sources are present. Upon integrating (4.5) over the subdomains of D where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\partial D} \Delta_{m,r,p,q}\nu_m[C(-\tau^{\mathrm{a}}_{p,q}, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau)]\,\mathrm{d}A\\
&\quad = \int_{\boldsymbol{x}\in D} [\partial_\tau C(\gamma^{\mathrm{b}}_{r,k} - \gamma^{\mathrm{a}}_{k,r}, v^{\mathrm{a}}_r, v^{\mathrm{b}}_k; \boldsymbol{x}, \tau) - \partial_\tau C(\kappa^{\mathrm{b}}_{p,q,i,j} - \kappa^{\mathrm{a}}_{i,j,p,q}, \tau^{\mathrm{a}}_{p,q}, \tau^{\mathrm{b}}_{i,j}; \boldsymbol{x}, \tau)]\,\mathrm{d}V\\
&\qquad + \int_{\boldsymbol{x}\in D} [C(f^{\mathrm{a}}_r, v^{\mathrm{b}}_r; \boldsymbol{x}, \tau) + C(-\tau^{\mathrm{a}}_{p,q}, h^{\mathrm{b}}_{p,q}; \boldsymbol{x}, \tau) - C(f^{\mathrm{b}}_r, v^{\mathrm{a}}_r; \boldsymbol{x}, \tau) - C(-\tau^{\mathrm{b}}_{p,q}, h^{\mathrm{a}}_{p,q}; \boldsymbol{x}, \tau)]\,\mathrm{d}V.
\end{aligned} \tag{4.6}$$

Equation (4.6) is the global form, for the domain D, of the time-convolution reciprocity theorem. Note that in the left-hand side the contributions from interfaces between different solids present in D have canceled and that the contributions from the boundaries of elastically impenetrable objects present in D have vanished in view of the boundary conditions stated in Section 3.

5. The reciprocity theorem of the time-correlation type

The reciprocity theorem of the time-correlation type follows upon considering the interaction quantity

$$\Delta_{m,r,p,q}[R(-\tau^{a}_{p,q}, v^{b}_{r}; \boldsymbol{x}, \tau)+R(-\tau^{b}_{p,q}, v^{a}_{r}; \boldsymbol{x}, -\tau)].$$

On account of (2.4) and (2.6), this interaction quantity is equivalent to

$$\Delta_{m,r,p,q}[C(-\tau^{a}_{p,q}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, v^{a}_{r}; \boldsymbol{x}, \tau)].$$

Using (3.1) and (3.2) for each of the two states we obtain

$$\Delta_{m,r,p,q}\partial_m C(-\tau^{a}_{p,q}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)=C(f^{a}_{r}-\dot{\Phi}^{a}_{r}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\tau^{a}_{p,q}, \bar{h}^{b}_{p,q}+\bar{\dot{e}}^{b}_{p,q}; \boldsymbol{x}, \tau) \tag{5.1}$$

and

$$\Delta_{m,r,p,q}\partial_m C(-\bar{\tau}^{b}_{p,q}, v^{a}_{r}; \boldsymbol{x}, \tau)=C(\bar{f}^{b}_{r}-\bar{\dot{\Phi}}^{b}_{r}, v^{a}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, h^{a}_{p,q}+\dot{e}^{a}_{p,q}; \boldsymbol{x}, \tau). \tag{5.2}$$

Now, in view of (3.5) and (3.6), we have

$$-C(\bar{\dot{\Phi}}^{b}_{r}, v^{a}_{r}; \boldsymbol{x}, \tau)-C(\dot{\Phi}^{a}_{r}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)=\partial_\tau C(\bar{\gamma}^{b}_{r,k}-\gamma^{a}_{k,r}, v^{a}_{r}, \bar{v}^{b}_{k}; \boldsymbol{x}, \tau) \tag{5.3}$$

and

$$C(-\tau^{a}_{p,q}, \bar{\dot{e}}^{b}_{p,q}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, \dot{e}^{a}_{p,q}; \boldsymbol{x}, \tau)=\partial_\tau C(\bar{\kappa}^{b}_{p,q,i,j}-\kappa^{a}_{i,j,p,q}, \tau^{a}_{p,q}, \bar{\tau}^{b}_{i,j}; \boldsymbol{x}, \tau), \tag{5.4}$$

where (2.7) has been used. Adding (5.2) to (5.1) and employing (5.3) and (5.4), we arrive at

$$\begin{aligned}
&\Delta_{m,r,p,q}\partial_m[C(-\tau^{a}_{p,q}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, v^{a}_{r}; \boldsymbol{x}, \tau)]\\
&\quad=\partial_\tau C(\bar{\gamma}^{b}_{r,k}-\gamma^{a}_{k,r}, v^{a}_{r}, \bar{v}^{b}_{k}; \boldsymbol{x}, \tau)+\partial_\tau C(\bar{\kappa}^{b}_{p,q,i,j}-\kappa^{a}_{i,j,p,q}, \tau^{a}_{p,q}, \bar{\tau}^{b}_{i,j}; \boldsymbol{x}, \tau)\\
&\qquad+C(f^{a}_{r}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\tau^{a}_{p,q}, \bar{h}^{b}_{p,q}; \boldsymbol{x}, \tau)+C(\bar{f}^{b}_{r}, v^{a}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, h^{a}_{p,q}; \boldsymbol{x}, \tau).
\end{aligned} \tag{5.5}$$

Equation (5.5) is the local form of the time-correlation reciprocity theorem. The first two terms in the right-hand side are representative for the differences in the properties of the solids present in the two states; they vanish at those locations where $\bar{\gamma}^{b}_{r,k}(\boldsymbol{x}, \tau)=\gamma^{a}_{k,r}(\boldsymbol{x}, \tau)$ and $\bar{\kappa}^{b}_{p,q,i,j}(\boldsymbol{x}, \tau)=\kappa^{a}_{i,j,p,q}(\boldsymbol{x}, \tau)$ for all $\tau\in\mathbb{R}$. In case the latter conditions hold, the two media are denoted as each other's time-reverse adjoints. Note in this respect that the time-reverse adjoint of a causal (effectual) medium is an effectual (causal) one. Upon integrating (5.5) over the subdomains of D where both sides are continuously differentiable, applying Gauss' divergence theorem to the resulting left-hand sides, and adding the results, we obtain

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\partial D}\Delta_{m,r,p,q}\nu_m[C(-\tau^{a}_{p,q}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, v^{a}_{r}; \boldsymbol{x}, \tau)]\,\mathrm{d}A\\
&\quad=\int_{\boldsymbol{x}\in D}[\partial_\tau C(\bar{\gamma}^{b}_{r,k}-\gamma^{a}_{k,r}, v^{a}_{r}, \bar{v}^{b}_{k}; \boldsymbol{x}, \tau)+\partial_\tau C(\bar{\kappa}^{b}_{p,q,i,j}-\kappa^{a}_{i,j,p,q}, \tau^{a}_{p,q}, \bar{\tau}^{b}_{i,j}; \boldsymbol{x}, \tau)]\,\mathrm{d}V\\
&\qquad+\int_{\boldsymbol{x}\in D}[C(f^{a}_{r}, \bar{v}^{b}_{r}; \boldsymbol{x}, \tau)+C(-\tau^{a}_{p,q}, \bar{h}^{b}_{p,q}; \boldsymbol{x}, \tau)+C(\bar{f}^{b}_{r}, v^{a}_{r}; \boldsymbol{x}, \tau)+C(-\bar{\tau}^{b}_{p,q}, h^{a}_{p,q}; \boldsymbol{x}, \tau)]\,\mathrm{d}V.
\end{aligned} \tag{5.6}$$

Equation (5.6) is the global form, for the domain D, of the time-correlation reciprocity theorem. Note that in the left-hand side the contributions from interfaces between different media present in D have canceled and that the contributions from the boundaries of elastically impenetrable objects present in D have vanished in view of the boundary conditions stated in Section 3.

6. Application to inverse problems

In this section we discuss the relevance of the reciprocity theorems (4.6) and (5.6) to inverse problems. In this respect, we distinguish between inverse source problems and inverse constituency problems. In an inverse source problem the aim is to reconstruct the volume source densities of strain rate and force of elastodynamic sources present in some inaccessible domain in space from the measured values of the emitted elastodynamic wave field in some other domain in space. The constitutive parameters of the solid in which the elastodynamic radiation takes place are assumed to be known. In an inverse constituency problem (also denoted as an inverse profiling or inverse scattering problem) the aim is to reconstruct the distribution of constitutive parameters in some inaccessible domain in space by irradiating the configuration by known sources located in the embedding and measuring the elastodynamic wave field response in some other domain in the embedding; the constitutive parameters of the embedding are known. The two types of problems will be discussed separately.

6.1. Inverse source problem

In the inverse source problem the elastodynamic wave field in state "a" is taken to be one that is radiated by the unknown source distributions $\{f_r^T, h_{p,q}^T\}$. Let $D^T \subset \mathbb{R}^3$ be their spatial support. The radiated wave field $\{-\tau_{p,q}^T, v_r^T\}$ is measured in some, accessible, observational domain $D^\Omega \subset \mathbb{R}^3$. The intersection of D^T and D^Ω is empty (Fig. 2). State "b" is taken to be a computational state, denoted as the "observational" one. The corresponding wave field $\{-\tau_{p,q}^\Omega, v_r^\Omega\}$ that would be radiated by known sources with distributions $\{f_r^\Omega, h_{p,q}^\Omega\}$ is computed and its interaction with the measured elastodynamic wave field in D^Ω is evaluated. In general, one could say that the introduction of the observational state is representative for the processing of the measured data. Since only the interaction in D^Ω is considered, it makes no sense to take the support of $\{f_r^\Omega, h_{p,q}^\Omega\}$ larger than D^Ω. Finally, the solid in the observational state is taken to be either the adjoint (for the application of (4.6)) or the time-reverse adjoint (for the application of (5.6)) of the one in which the unknown sources radiate.

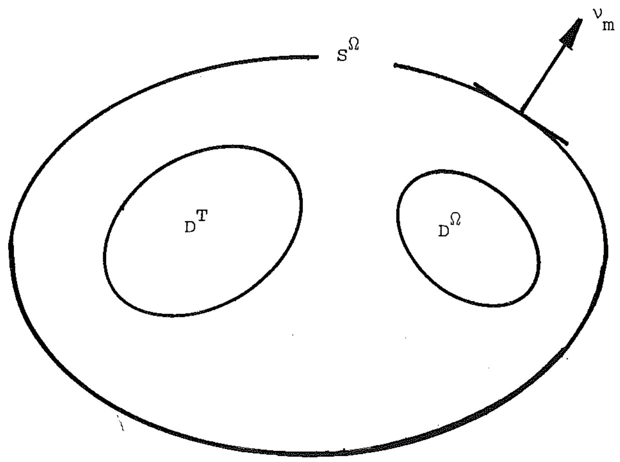

Fig. 2. Configuration illustrative for the inverse source problem: unknown acoustic sources radiate in D^T; the elastodynamic wave field is measured in D^Ω and on S^Ω.

The reciprocity relations (4.6) and (5.6) are now applied to the domain interior to the closed surface S^{Ω} that is taken such that D^T and D^{Ω} are located in its interior. Then, (4.6) leads to

$$\int_{x \in D^T} [-C(f_r^T, v_r^{\Omega}; x, \tau) + C(-\tau_{p,q}^{\Omega}, h_{p,q}^T; x, \tau)] \, dV$$
$$= \int_{x \in D^{\Omega}} [-C(f_r^{\Omega}, v_r^T; x, \tau) + C(-\tau_{p,q}^T, h_{p,q}^{\Omega}; x, \tau)] \, dV$$
$$- \int_{x \in S^{\Omega}} \Delta_{m,r,p,q} \nu_m [C(-\tau_{p,q}^T, v_r^{\Omega}; x, \tau) - C(-\tau_{p,q}^{\Omega}, v_r^T; x, \tau)] \, dA, \tag{6.1}$$

and (5.6) leads to

$$\int_{x \in D^T} [-C(f_r^T, \bar{v}_r^{\Omega}; x, \tau) - C(-\bar{\tau}_{p,q}^{\Omega}, h_{p,q}^T; x, \tau)] \, dV$$
$$= \int_{x \in D^{\Omega}} [C(\bar{f}_r^{\Omega}, v_r^T; x, \tau) + C(-\tau_{p,q}^T, \bar{h}_{p,q}^{\Omega}; x, \tau)] \, dV$$
$$- \int_{x \in S^{\Omega}} \Delta_{m,r,p,q} \nu_m [C(-\tau_{p,q}^T, \bar{v}_r^{\Omega}; x, \tau) + C(-\bar{\tau}_{p,q}^{\Omega}, v_r^T; x, \tau)] \, dA. \tag{6.2}$$

In (6.1) and (6.2) the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on S^{Ω}. A solution to the inverse source problem is now commonly constructed by taking for $\{f_r^{\Omega}, h_{p,q}^{\Omega}\}$ a sequence of N linearly independent distributions with spatial support D^{Ω} and fixed, preferably broadband, time behavior (for example, an impulse). The corresponding sequence of elastodynamic wave field distributions $\{-\tau_{p,q}^{\Omega}, v_r^{\Omega}\}$ is computed. Next, the unknown source distributions $\{f_r^T, h_{p,q}^T\}$ are expanded into an appropriate sequence of M linearly independent space-time expansion functions with spatial support D^T; the corresponding expansion coefficients are unknown. Substitution of the results in (6.1) and (6.2), and the evaluation of the relevant integrals leads to systems of linear algebraic equations with the source expansion coefficients as unknowns. When $M = N$, the system can be solved, unless the pertaining matrix of coefficients is singular. However, even if this matrix is non-singular, it turns out to be ill-conditioned in all practical cases. Therefore, one usually takes $M > N$, and a best fit of the expanded source distributions is obtained by the application of minimization techniques.

At this point, some more must be said about the role of S^{Ω}. In practice, one is as a rule interested only in causal media. Then, in the application of (6.1), it is advantageous to choose the wave fields causal as well. Given the fact that S^{Ω} surrounds all sources, the integral over S^{Ω} can be replaced by an integral over any sphere S_{Δ} with radius Δ and center at the origin such that S_{Δ} surrounds S^{Ω}. (This follows from the application of (4.6) to the domain in between S_{Δ} and S^{Ω}.) However, for sufficiently large values of Δ, the causal wave field on S_{Δ} is zero since it propagates with a finite maximum speed away from the sources. Hence, under these circumstances, the surface integral in the right-hand side of (6.1) vanishes. A similar argument does not apply to the surface integral in the right-hand side of (6.2) since in (6.2) effectual (or anticausal) wave fields are involved in all cases. This difference in the roles of the surface integrals in (6.1) and (6.2) has been pointed out by Bojarski [3].

6.2. Inverse constituency problem

In the inverse constituency problem the elastodynamic wave field in state "a" is taken to be the one that irradiates the configuration. Let $D^i \subset \mathbb{R}^3$ be the spatial support of the irradiating sources with known distributions $\{f^i_r, h^i_{p,q}\}$ and let the corresponding elastodynamic wave field be $\{-\tau^i_{p,q}, v^i_r\}$. This wave field is measured in some accessible, observational domain $D^\Omega \subset \mathbb{R}^3$. Let, further, $D^s \subset \mathbb{R}^3$ be the (inaccessible) domain in which the constitutive parameters are unknown. The intersections of D^s and D^i and of D^s and D^Ω are empty; D^i and D^Ω may, however, have points in common, or may even completely coincide (Fig. 3). State "b" is taken to be a computational state, denoted as the "observational" state. The corresponding wave field $\{-\tau^\Omega_{p,q}, v^\Omega_r\}$ that would be emitted by known elastodynamic sources with distributions $\{f^\Omega_r, h^\Omega_{p,q}\}$ in the known solid with the constitutive parameters $\{\kappa^\Omega_{p,q,i,j}, \gamma^\Omega_{r,k}\}$ of the adjoint (for the application of (4.6)) or the time-reverse adjoint (for the application of (5.6)) of the known embedding is computed and its interaction with the measured wave field in D^Ω is evaluated. Since only the interaction in D^Ω is considered, it makes no sense to take the support of $\{f^\Omega_r, h^\Omega_{p,q}\}$ larger than D^Ω. The unknown constitutive parameters of D^s are denoted by $\{\kappa^s_{i,j,p,q}, \gamma^s_{k,r}\}$, D^s being the support of the differences

$$\{\kappa^s_{i,j,p,q} - \kappa^\Omega_{p,q,i,j}, \gamma^s_{k,r} - \gamma^\Omega_{r,k}\} \quad \text{and} \quad \{\kappa^s_{i,j,p,q} - \bar{\kappa}^\Omega_{p,q,i,j}, \gamma^s_{k,r} - \bar{\gamma}^\Omega_{r,k}\}$$

for the application of (4.6) and (5.6) respectively.

The reciprocity relations (4.6) and (5.6) are now applied to the domain interior to the surface S^Ω that is taken such that D^i, D^s and D^Ω are located in its interior. Then (4.6) leads to

$$\begin{aligned}
&\int_{x\in D^s} [-C(f^s_r, v^\Omega_r; x, \tau) + C(-\tau^\Omega_{p,q}, h^s_{p,q}; x, \tau)]\,\mathrm{d}V \\
&= \int_{x\in D^i} [C(f^i_r, v^\Omega_r; x, \tau) - C(-\tau^\Omega_{p,q}, h^i_{p,q}; x, \tau)]\,\mathrm{d}V \\
&\quad + \int_{x\in D^\Omega} [-C(f^\Omega_r, v^i_r; x, \tau) + C(-\tau^i_{p,q}, h^\Omega_{p,q}; x, \tau)]\,\mathrm{d}V \\
&\quad - \int_{x\in S^\Omega} \Delta_{m,r,p,q}\nu_m [C(-\tau^i_{p,q}, v^\Omega_r; x, \tau) - C(-\tau^\Omega_{p,q}, v^i_r; x, \tau)]\,\mathrm{d}A,
\end{aligned} \tag{6.3}$$

in which

$$h^s_{p,q} = \partial_\tau C(\kappa^\Omega_{i,j,p,q} - \kappa^s_{p,q,i,j}, -\tau^i_{i,j}; x, \tau) \tag{6.4}$$

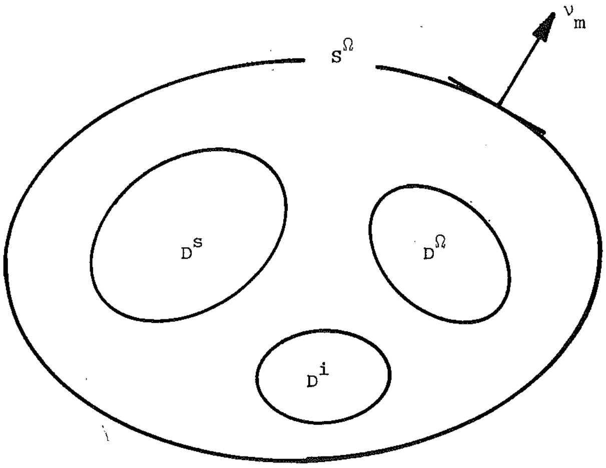

Fig. 3. Configuration illustrative for the inverse constituency problem: known acoustic sources in D^i irradiate the contrasting domain D^s with unknown properties; the elastodynamic wave field is measured in D^Ω and on S^Ω.

is the equivalent contrast volume source density of strain rate in D^s, and

$$f_r^s = \partial_\tau C(\gamma_{k,r}^{\Omega} - \gamma_{r,k}^{s}, v_k^i; \boldsymbol{x}, \tau) \tag{6.5}$$

is the equivalent contrast volume source density of force in D^s. In the same way, (5.6) leads to

$$\begin{aligned}
&\int_{\boldsymbol{x}\in D^s} [-C(f_r^s, \bar{v}_r^{\Omega}; \boldsymbol{x}, \tau) - C(-\bar{\tau}_{p,q}^{\Omega}, h_{p,q}^s; \boldsymbol{x}, \tau)]\, \mathrm{d}V \\
&\quad = \int_{\boldsymbol{x}\in D^i} [C(f_r^i, \bar{v}_r^{\Omega}; \boldsymbol{x}, \tau) + C(-\bar{\tau}_{p,q}^{\Omega}, h_{p,q}^i; \boldsymbol{x}, \tau)]\, \mathrm{d}V \\
&\quad\quad + \int_{\boldsymbol{x}\in D^{\Omega}} [C(\bar{f}_r^{\Omega}, v_r^i; \boldsymbol{x}, \tau) + C(-\tau_{p,q}^i, \bar{h}_{p,q}^{\Omega}; \boldsymbol{x}, \tau)]\, \mathrm{d}V \\
&\quad\quad - \int_{\boldsymbol{x}\in S^{\Omega}} \Delta_{m,r,p,q}\nu_m [C(-\tau_{p,q}^i, \bar{v}_r^{\Omega}; \boldsymbol{x}, \tau) + C(-\bar{\tau}_{p,q}^{\Omega}, v_r^i; \boldsymbol{x}, \tau)]\, \mathrm{d}A,
\end{aligned} \tag{6.6}$$

in which

$$h_{p,q}^s = \partial_\tau C(\bar{\kappa}_{i,j,p,q}^{\Omega} - \kappa_{p,q,i,j}^{s}, -\tau_{i,j}^i; \boldsymbol{x}, \tau) \tag{6.7}$$

is the equivalent contrast volume source density of strain rate in D^s, and

$$f_r^s = \partial_\tau C(\bar{\gamma}_{k,r}^{\Omega} - \gamma_{r,k}^{s}, v_k^i; \boldsymbol{x}, \tau) \tag{6.8}$$

is the equivalent contrast volume source density of force in D^s.

In (6.3) and (6.6), the left-hand sides contain the unknown quantities, while the right-hand sides are known provided that the necessary measurements and evaluations are also carried out on S^{Ω}. The easiest way to address the inverse constituency problem is to consider it as an inverse source problem for the quantities $\{h_{p,q}^s, f_r^s\}$. Once values for these have been obtained, the solution of the forward or direct source problem with known values of $\{h_{p,q}^i, f_r^i\}$ and $\{h_{p,q}^s, f_r^s\}$ yields the values of $\{-\tau_{p,q}^i, v_r^i\}$ in D^s and since $\{\kappa_{i,j,p,q}^{\Omega}, \gamma_{k,r}^{\Omega}\}$ are known, the temporal deconvolution of either (6.4) and (6.5) or (6.7) and (6.8) yields the values of $\{\kappa_{p,q,i,j}^s, \gamma_{r,k}^s\}$. As to the role of the surface integrals over S^{Ω} in the right-hand sides of (6.3) and (6.6), the same remarks as for the inverse source problem apply.

To conclude our investigation, we want to emphasize that the uniqueness and the existence of solutions to both the inverse source and the inverse constituency problem are, for the larger part, at the moment open questions.

7. Conclusion

Time-domain reciprocity theorems for the elastodynamic wave field in linear, time-invariant, and locally reacting media have been derived via a full space-time method. Inhomogeneity, anisotropy, and relaxation effects of the solid are included. One of the theorems is of the time-convolution type, the other of the time-correlation type. The application of the two theorems to inverse source and inverse constituency problems has been discussed.

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