A FINITE-ELEMENT METHOD FOR COMPUTING THREE-DIMENSIONAL ELECTROMAGNETIC FIELDS IN INHOMOGENEOUS MEDIA

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Abstract - A finite-element method is presented that is particularly suited for the computer modeling of three-dimensional electromagnetic fields in inhomogeneous media. It employs a new type of linear vectorial expansion functions. Across an interface where the constitutive coefficients are discontinuous, they have the following properties: (1) the continuity of the tangential components of the electric and the magnetic field strengths is exactly preserved, (2) the normal component of the electric and the magnetic field strengths are allowed to jump and (3) the electric and the magnetic fluxes are continuous within the pertaining degree of approximation. The system of equations from which the expansion coefficients are obtained is generated by applying a Galerkin-type weighted-residual method. Numerical experiments are described that illustrate the efficiency of our elements, and the computational costs of the method.

INTRODUCTION

Because of its flexibility, the finite-element method seems to be the most suitable one to compute electromagnetic fields in inhomogeneous media. Presently available programs [1, 2] are, however, limited to media having a low contrast and to simple geometries. Studying the expansion functions that are used it is easily shown that they do not permit the fluxes of the electric and the magnetic field to be continuous at an interface where the constitutive coefficients are discontinuous. In the present paper we present a type of element that exactly accounts for the continuity of the tangential components of the electric and the magnetic field strengths across interfaces where the constitutive coefficients are discontinuous, and that permit the fluxes of the electric and the magnetic field to be continuous at those interfaces. Elements that exactly preserve the continuity of the tangential components have been proposed by Nédelec [3], and have been used by Bossavit and Vérité [4, 5] for solving eddy-current problems. The disadvantage of their elements, however, is that they are not consistently of a certain degree of approximation. The elements used by Bossavit and Vérité, for instance, permit a first-order interpolation of the field in the interior of the tetrahedron in which they apply, but yield only an approximation of order zero along the edges of this tetrahedron. Our elements are consistently linear, i.e. they yield a linear approximation of the field both inside each tetrahedron and along its edges and faces.

THE EXPANSION FUNCTIONS

The geometrical domain over which the finite-element method is applied, is subdivided into adjoining tetrahedra. In this section, we present the type of consistently linear vectorial expansion functions that will be used over each tetrahedron.

Using the right-handed, orthogonal Cartesian coordinates $\{x,y,z\}$, the position vectors of the vertices of a tetrahedron T are denoted by $\{\underline{r}_0,\underline{r}_1,\underline{r}_2,\underline{r}_3\}$. The outwardly directed vectorial areas of the faces of T are then given by (Fig. 1)

$$\underline{A}_0 = (\underline{r}_1 \times \underline{r}_2 + \underline{r}_2 \times \underline{r}_3 + \underline{r}_3 \times \underline{r}_1)/2, \text{ etc.} \tag{1}$$

The volume of T is given by

$$V = ((\underline{r}_1 - \underline{r}_0) \times (\underline{r}_2 - \underline{r}_0))\cdot(\underline{r}_3 - \underline{r}_0)/6. \tag{2}$$

Let

$$\underline{r}_b = (\underline{r}_0 + \underline{r}_1 + \underline{r}_2 + \underline{r}_3)/4 \tag{3}$$

be the position vector of the barycenter of T. Then the linear scalar function of position $\phi_i(\underline{r})$ that equals unity when $\underline{r} = \underline{r}_i$ (i=0,1,2,3) and that equals zero in the remaining three vertices of the tetrahedron can be written as

$$\phi_i(\underline{r}) = 1/4 - (\underline{r} - \underline{r}_b)\cdot\underline{A}_i/3V. \tag{4}$$

Since $\underline{r} = \sum_{i=0}^{3} \phi_i(\underline{r})\underline{r}_i$, with $\sum_{i=0}^{3} \phi_i(\underline{r}) = 1$, $\{\phi_0,\phi_1,\phi_2,\phi_3\}$ are nothing but the barycentric coordinates in T. Using (4) our vectorial expansion funtions $\underline{w}_{i,j}$ are taken to be

$$\underline{w}_{i,j}(\underline{r}) = -\phi_i(\underline{r})\underline{A}_j/3V, \quad (i,j=0,1,2,3 \quad i\neq j). \tag{5}$$

It is easily verified that $\{\underline{w}_{i,j}\}$ has the following properties: (a) $\underline{w}_{i,j}$ is a linear vector function of position in T, (b) the projection of it on an edge vanishes on all edges of T apart from the one joining the vertices $\underline{r}_i$ and $\underline{r}_j$ and (c) $\underline{w}_{i,j}$ varies linearly along the latter edge such that $\underline{w}_{i,j}(\underline{r}_j) = \underline{0}$, while it is normalized such that

$$\underline{w}_{i,j}(\underline{r}_i)\cdot(\underline{r}_j - \underline{r}_i) = 1. \tag{6}$$

The divergence and the curl of $\underline{w}_{i,j}$ are obtained as

$$\underline{\nabla}\cdot\underline{w}_{i,j}(\underline{r}) = \underline{A}_i\cdot\underline{A}_j/(3V)^2 \tag{7}$$

and

$$\underline{\nabla} \times \underline{w}_{i,j}(\underline{r}) = \underline{A}_i \times \underline{A}_j/(3V)^2, \tag{8}$$

respectively. Due to the normalization (6), the expansion functions that originate from the same vertex at a common edge of two or more adjoining tetrahedra have the same tangential behavior along this edge and over a common face, if present. In this way, expressing the electric and the magnetic field strengths in terms of $\{\underline{w}_{i,j}\}$, the continuity of the tangential component of these fields along interfaces is guaranteed. Observe that two expansion functions, viz. $\underline{w}_{i,j}$ and $\underline{w}_{j,i}$ do originate from each edge as compared with the single one in the expansion used by Bossavit and Vérité (Fig. 2). It can easily be shown that their expansion functions are, apart from a constant factor, given by $\underline{w}_{i,j} - \underline{w}_{j,i}$.

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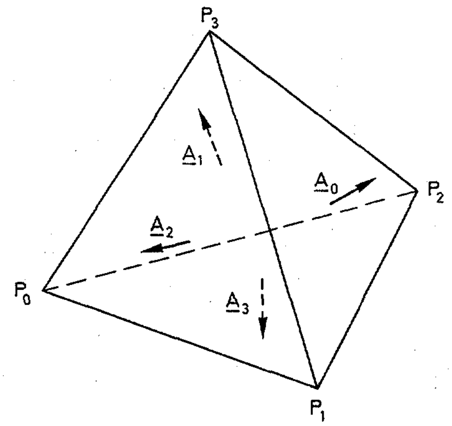

Fig. 1. Tetrahedron T with outwardly directed vectorial areas of its faces.

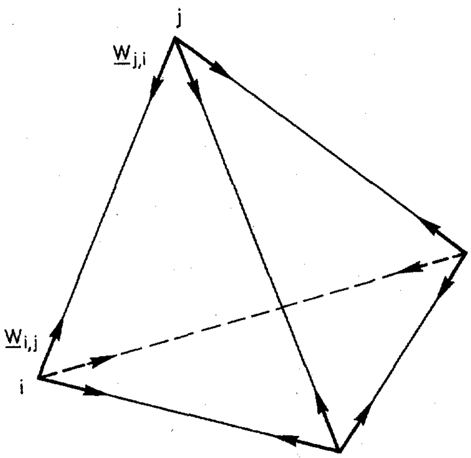

Fig. 2. Degrees of freedom in a tetrahedron.

When $\underline{r}\in T$, any vectorial quantity whose tangential components are to be continuous at interfaces, in our case the electric field $\underline{E}$ and the magnetic field strength $\underline{H}$, is expanded as

$$\underline{E} = \sum_{i=0}^{3} \sum_{\substack{j=0 \\ i\neq j}}^{3} e_{i,j}\, \underline{w}_{i,j}, \tag{9}$$

and

$$\underline{H} = \sum_{i=0}^{3} \sum_{\substack{j=0 \\ i\neq j}}^{3} h_{i,j}\, \underline{w}_{i,j}, \tag{10}$$

where $e_{i,j}$ and $h_{i,j}$ are the unknown expansion coefficients. In view of the normalization (6) of the vectorial expansion functions, the quantities $\{e_{i,j}\}$ and $\{h_{i,j}\}$ can be considered as the components of $\underline{E}(\underline{r}_i)$ and $\underline{H}(\underline{r}_i)$ along a set of base vectors formed by the edges $(\underline{r}_j - \underline{r}_i)$ that have $\underline{r}_i$ as common vertex. Note that $\{-\underline{A}_j/3V\}$ are the set of base vectors reciprocal to $\{\underline{r}_j - \underline{r}_i\}$.

THE METHOD OF WEIGHTED RESIDUALS

In the complex-frequency or s-domain the electromagnetic field equations for an isotropic medium are written in the form

$$\underline{\nabla} \times \underline{H} - Y_T\, \underline{E} = \underline{J}, \tag{11}$$

$$\underline{\nabla} \times \underline{E} + Z_L\, \underline{H} = -\underline{K}, \tag{12}$$

where $Y_T = \sigma + s\varepsilon$ and $Z_L = s\mu$ (σ = conductivity, ε = permittivity, μ = permeability). At a surface S of discontinuity in the properties of the medium we have the continuity conditions

$$\left.\begin{matrix} \underline{n} \times \underline{E} \\ \underline{n} \times \underline{H} \end{matrix}\right\} \quad \text{continuous across S,} \tag{13}$$

and

$$\left.\begin{matrix} \underline{n}\cdot Y_T\, \underline{E} \\ \underline{n}\cdot Z_L\, \underline{H} \end{matrix}\right\} \quad \text{continuous across S.} \tag{14}$$

In (13) and (14), $\underline{n}$ denotes a unit vector along the normal to S.

In the method of weighted residuals, both $\underline{E}$ and $\underline{H}$ are, in each tetrahedron, expanded using (9) and (10). Subsequently, (11) and (12) are multiplied scalarly by some weighting function and integrated over a tetrahedral domain.

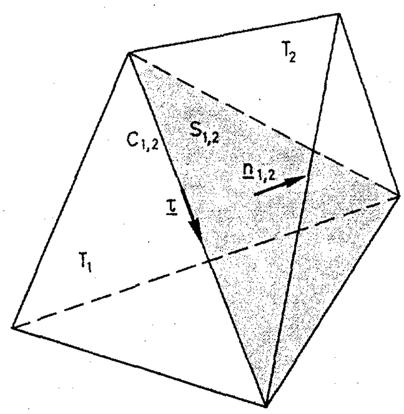

Fig. 3. Two adjacent tetrahedra having a triangular interface in common.

In the Galerkin method we choose $\{\underline{w}_{i,j}\}$ as the weighting functions. With this choice, and assuming the medium properties, as well as the source densities, to be constant in the interior of each tetrahedron, all integrations can be carried out analytically. As a result, we obtain a system of linear relations between the local expansion coefficients. In the finite-element method [6] these relations are properly combined to yield the global system of equations to be satisfied.

We shall now show that, within our degree of approximation, the continuity of the fluxes associated with (14) is also satisfied. To do this we study the solution in two adjacent tetrahedra T_1 and T_2 having a triangular interface $S_{1,2}$ (Fig. 3) in common. In view of the expansion functions, equations (13) are exactly satisfied. Both in T_1 and T_2, we multiply (11) scalarly by the unit vector $\underline{n}_{1,2}$ that is normal to $S_{1,2}$ and integrate the resulting expressions over $S_{1,2}$. Then, applying Stokes's theorem we obtain

$$\oint_{C_{1,2}} \underline{H}\cdot\underline{\tau}ds = \iint_{S_{1,2}} \underline{n}_{1,2}\cdot(Y_{T,1}\underline{E}_1 + \underline{J}_1)dA$$
$$= \iint_{S_{1,2}} \underline{n}_{1,2}\cdot(Y_{T,2}\underline{E}_2 + \underline{J}_2)dA, \qquad (15)$$

where $C_{1,2}$ is the closed boundary curve of $S_{1,2}$ and $\underline{\tau}$ is the unit vector along the tangent to $C_{1,2}$ in the direction of circulation that forms a right-handed system with $\underline{n}_{1,2}$. If the approximations to $\underline{E}$ and $\underline{H}$ would exactly satisfy (11) and (12), the difference between the two right-handed sides of (15) would vanish exactly since $\underline{H}\cdot\underline{\tau}$ is exactly continuous through the choice of our expansion functions. Now, (11) and (12) are satisfied up to order $O(d^2)$, where d is the maximum dimension of a tetrahedron, and hence (15), and consequently the continuity of the fluxes, are expected to be satisfied up to order $O(d^2)$. In the same way the continuity of the magnetic flux can be proven starting from (12).

RESULTS FOR A TWO-DIMENSIONAL CONFIGURATION

In order to investigate the usefulness of our elements we have also written computer codes for the simpler two-dimensional, electromagnetic fields in a cylindrical configuration whose properties are independent of z. The two-dimensional fields then separate into an E-polarized part in which $\{E_z,H_x,H_y\}$ occur and an H-polarized part in which $\{H_z,E_x,E_y\}$ occur. The method of solution was applied to the case where in the domain $0<x<1$, $0<y<1$ and in a homogeneous medium with $\varepsilon=\varepsilon_0$, $\mu=\mu_0$, $\sigma=0$, the field components of the time-harmonic field with $s=j\omega$, $\omega=4\pi*10^8$, were prescribed as

$$\underline{E} = \sin(\pi x)\sin(\pi y)\underline{i}_z,$$
$$\underline{H} = (j\pi/\omega\mu_0)\sin(\pi x)\cos(\pi y)\underline{i}_x - (j\pi/\omega\mu_0)\cos(\pi x)\sin(\pi x)\underline{i}_y, \qquad (16)$$

the expressions further entail

$$\underline{J} = J_0\sin(\pi x)\sin(\pi y)\underline{i}_z,\quad \underline{K} = \underline{0}, \qquad (17)$$

with $J_0 = j(2\pi^2 - \omega^2\varepsilon_0\mu_0)/\omega\mu_0$. The boundary condition $E_z=0$ was specified. The region was divided into isoceles rectangular triangles, two triangles occupying a square region of dimension h × h. The system of equations was solved using a direct method. In Table 1, the relative error in the numerical results near the center of the domain and the computation time are given for Nédelec elements and for our elements. Comparing the relative errors while taking into account the computation times it is clear that our elements yield more accurate results with less computational effort. Note that, for the problem at hand, we have $\underline{\nabla}\cdot\underline{E}=0$ and $\underline{\nabla}\cdot\underline{H}=0$. This suggests that using expansion functions with zero divergence, as Nédelec's are, would be preferable. Our results prove the contrary. A series of other numerical experiments has been carried out. They proved the efficiency of our expansion functions for computing fields in inhomogeneous media. In one of those experiments we have taken into account a linear variation of the properties of the media and the source density distribution over a triangle. It turned out that these linear variations did not improve the accuracy of the results as compared with the approximation by a constant.

Table 1. Relative errors in % and computation time T for Nédelec elements and our elements for a test case (E-polarized electromagnetic field in a domain of 1m×1m), symmetric mesh, E-field error.

grid size h(m)	Nédelec elements rel.err(%)	T(s)	Our elements rel.err(%)	T(s)
1/4	50	3.0	1.7	3.5
1/8	10	4.5	.25	8.9
1/16	2.5	19.7	-	-

RESULTS FOR A THREE-DIMENSIONAL CONFIGURATION

As a first test of our three-dimensional code we have applied it to the case where in the domain $0<x<1$, $0<y<1$, $0<z<0.1$ and in a homogeneous medium with $\varepsilon=\varepsilon_0$, $\mu=\mu_0$, $\sigma=0$, the field components of the time-harmonic field with $s=j\omega$, $\omega=4\pi*10^8$, were prescribed as is given in (16) and (17), the remaining components of the field being zero. The boundary condition $\underline{n}\times\underline{E}=\underline{0}$ was used, but any other combination of boundary conditions consistent with (16) could have been used. The region was divided in subregions of dimension h × h × 0.1m, each of these subregions being subdivided into six tetrahedra. In Table 2 we present, as a function of the grid size h, some results with regard to the number of unknowns N, the computation time, the storage requirements and the relative error in the final result. The relative error was studied at a diagonal plane in the configuration. In such a plane the solution has the most rapid variation in space (one period of a sine) and hence our results for the relative error can be considered as a worst case result. All computations have been carried out on an Amdahl 470V/7B computer using the SEPRAN finite-element package [7].

Table 2. Computational requirements and maximum relative error in the results for a three-dimensional test problem.

grid size h(m)	number of unknowns	computation time T(s)	storage (Mbyte)	rel.err. (%)
1/2	112	2	1.0	30
1/4	448	15	1.5	12
1/8	1792	215	5.5	5

CONCLUSION

We have presented a new type of consistently linear vectorial expansion function that exactly accounts for the continuity of both the tangential components of the vector functions approximated across interfaces and the continuity of the normal component of the fluxes. In a numerical experiment, a two-dimensional version of these functions proved to yield very accurate results within a relatively small computational effort. The three-dimensional version of our code also produces excellent results but in this case the computational costs, especially when using a direct method for solving the system of equations, are quite considerable. In order to reduce the computational costs, iterative techniques will have to be used for the latter. An additional advantage of the method described is that, because of the nature of the expansion functions, it is ideally suited for a combination of it with the Boundary Element Method.

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