Elektronica en Telecommunicatie 6

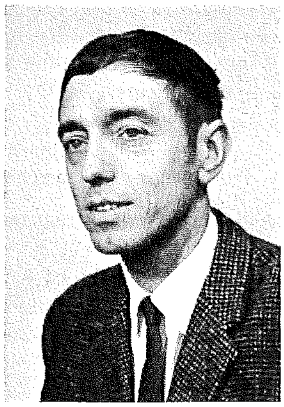
Gemeenschappelijke publikatie van de Sectie voor Telecommunicatietechniek van het KIvI en het Nederlands Elektronica- en Radiogenootschap.
Redactie-adres: Prinsessegracht 23, Den Haag.

Symposium Antennas

621.396.67:538.56

II. Reciprocity in Antenna Theory

by prof. dr. ir. A. T. de Hoop, Department of Electrical Engineering, Delft University of Technology



Synopsis: A reciprocity relation between the transmitting and the receiving properties of an antenna is given. In the transmitting situation a certain part of the antenna, called 'source domain', is capable of carrying external currents, both of the electric and the magnetic type. In the receiving situation a plane electromagnetic wave is incident upon the antenna system.

In particular, the reciprocity relation is applied to a microwave antenna system in which the feeding waveguide operates in more than a single mode.

1. Introduction

One of the basic theorems in the electromagnetic theory of antennas is a reciprocity relation between the transmitting and the receiving properties of an antenna. The customary form of this reciprocity relation (see, for example, [1]) applies to two different antennas, a finite distance apart, each of them playing in turn the role of transmitting antenna or receiving antenna. In technical applications, however, one frequently employs the reciprocity theorem to relate the radiation pattern of a transmitting antenna to the pattern of the same antenna when it is receiving an incident plane wave. Clearly, for the latter problem a reciprocity theorem applicable to a single antenna is required. Now, such a reciprocity theorem can be obtained from the aforementioned reciprocity theorem applicable to two separate antennas by letting the mutual distance between the antennas become very large and considering the limiting form of the relation between the electromagnetic field quantities involved. However, a reciprocity theorem applicable to a single antenna can also be obtained directly, without having to resort to the corresponding problem for two separate antennas; this has been shown by the author in an earlier paper [2].

In the present paper the relevant reciprocity theorem, and the conditions under which it holds, are briefly stated, whereupon the application to a microwave antenna system is worked out in detail.

2. Description of the configuration

The antenna system under consideration occupies a bounded domain V in space; the boundary of V is a sufficiently regular closed surface S. The Cartesian co-ordinates of a point in space are denoted by x, y and z; the time variable is denoted by t. The position vector is denoted by $\mathbf{r}$. The electromagnetic fields occurring in the transmitting as well as in the receiving situation are assumed to vary sinusoidally in time with the same angular

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Lectures delivered at the symposium on antennas, organized by the Nederlands Elektronica- en Radiogenootschap, on May 18, 1971 at Delft University of Technology, Department of Electrical Engineering. See no. 1 in 'De Ingenieur' 1972, nr. 20, pp. ET 51 ... ET 68. For the announcement see 'De Ingenieur', 1971, nr. 19, p. A 336.

frequency ω. The complex representation of the field vectors is used; in the formulas the complex time factor $\exp(-i\omega t)$, common to all field components, is omitted.

The antenna consists of a (partly lossy) medium whose electromagnetic behaviour is assumed to be linear. The properties of the medium may suddenly change when crossing a (bounded) surface; across such a discontinuity surface the tangential parts of both the electric and the magnetic field vector are continuous. Other parts of the antenna system may consist of conducting surfaces. These surfaces are assumed to be electrically perfectly conducting; on them the tangential part of the electric field vector vanishes.

In the transmitting situation a subdomain V_{source} of V is capable of carrying 'external' currents, both of electric and magnetic type. These external currents represent the sources through which power of nonelectromagnetic origin can be delivered to the system. The boundary of V_{source} is a sufficiently regular closed surface S_{source}. In the receiving situation no external currents are present in the antenna system.

The medium outside the antenna system is assumed to be linear, homogeneous, isotropic, and lossless (which includes the case of free space) with real scalar permittivity ε_0 and real scalar permeability μ_0.

In the following $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$ and $\mathbf{B}$ denote the space and frequency dependent complex representations of the electric field vector, the magnetic field vector, the electric flux density and the magnetic flux density, respectively. The superscripts 'T' and 'R' are used to denote the transmitting and the receiving situation, respectively.

3. The antenna as a transmitting system

In the transmitting situation the antenna carries time harmonic external currents. Let $\mathbf{J}^{\text{T}}$ be the volume density of external electric currents and let $\mathbf{K}^{\text{T}}$ be the volume density of external magnetic currents; $\mathbf{J}^{\text{T}}$ and $\mathbf{K}^{\text{T}}$ differ from zero in V_{source} only. In V the electromagnetic field vectors satisfy the inhomogeneous Maxwell equations:

$$\operatorname{rot} \mathbf{H}^{\text{T}} + i\omega \mathbf{D}^{\text{T}} = \mathbf{J}^{\text{T}}, \tag{3.1}$$

$$\operatorname{rot} \mathbf{E}^{\text{T}} - i\omega \mathbf{B}^{\text{T}} = -\mathbf{K}^{\text{T}} \tag{3.2}$$

and the constitutive equations which express $(\mathbf{D}^{\text{T}}, \mathbf{B}^{\text{T}})$ linearly in terms of $(\mathbf{E}^{\text{T}}, \mathbf{H}^{\text{T}})$. Further, the tangential parts of both $\mathbf{E}^{\text{T}}$ and $\mathbf{H}^{\text{T}}$ are continuous across a surface of discontinuity in material properties. In the domain outside the antenna system $\mathbf{E}^{\text{T}}$ and $\mathbf{H}^{\text{T}}$ satisfy the homogeneous Maxwell equations:

$$\operatorname{rot} \mathbf{H}^{\text{T}} + i\omega\varepsilon_0 \mathbf{E}^{\text{T}} = \mathbf{0}, \tag{3.3}$$

$$\operatorname{rot} \mathbf{E}^{\text{T}} - i\omega\mu_0 \mathbf{H}^{\text{T}} = \mathbf{0}. \tag{3.4}$$

In addition, the transmitted field satisfies the radiation conditions (Cf. [3]). As a consequence the following representation holds at large distances from the antenna system:

$$\mathbf{E}^{\text{T}}(\mathbf{r}_{\text{P}}) \simeq \mathbf{F}^{\text{T}}(\hat{\mathbf{r}}_{\text{P}}) \exp(ik_0 r_{\text{P}})/4\pi r_{\text{P}}, \tag{3.5}$$

$$\mathbf{H}^{\text{T}}(\mathbf{r}_{\text{P}}) \simeq (\varepsilon_0/\mu_0)^{\frac{1}{2}} (\hat{\mathbf{r}}_{\text{P}} \times \mathbf{F}^{\text{T}}) \exp(ik_0 r_{\text{P}})/4\pi r_{\text{P}}, \tag{3.6}$$

in which:

$$k_0 \stackrel{\text{def}}{=} \omega\, (\varepsilon_0\mu_0)^{\frac{1}{2}} = 2\pi/\lambda_0, \tag{3.7}$$

where λ_0 is the wavelength, $\hat{\mathbf{r}}_{\text{P}} \stackrel{\text{def}}{=} \mathbf{r}_{\text{P}}/r_{\text{P}}$ is the unit vector in the direction of observation and r_{P} is the distance from the origin of the coordinate system to the point of observation P. The vectorial amplitude radiation characteristic $\mathbf{F}^{\text{T}}$ of the antenna system only depends upon the direction of observation $\hat{\mathbf{r}}_{\text{P}}$ and is transverse with respect to the direction of propagation of the expanding spherical wave generated by the antenna, i.e. $\hat{\mathbf{r}}_{\text{p}} \cdot \mathbf{F}^{\text{T}} = 0$. Further, $\mathbf{F}^{\text{T}}$ can be expressed in terms of the value the tangential parts of $\mathbf{E}^{\text{T}}$ and $\mathbf{H}^{\text{T}}$ have on the boundary surface S of the antenna system (for details, see [2]).

4. The antenna as a receiving system

In the receiving situation a time harmonic plane electromagnetic wave is incident upon the antenna system. Since in the receiving situation the antenna carries no external currents, the electromagnetic field vectors satisfy in V the homogeneous Maxwell equations:

$$\operatorname{rot} \mathbf{H}^{\text{R}} + i\omega \mathbf{D}^{\text{R}} = \mathbf{0}, \tag{4.1}$$

$$\operatorname{rot} \mathbf{E}^{\text{R}} - i\omega \mathbf{B}^{\text{R}} = \mathbf{0}, \tag{4.2}$$

and the constitutive equations which express $(\mathbf{D}^{\text{R}}, \mathbf{B}^{\text{R}})$ linearly in terms of $(\mathbf{E}^{\text{R}}, \mathbf{H}^{\text{R}})$. Further, the tangential parts of both $\mathbf{E}^{\text{R}}$ and $\mathbf{H}^{\text{R}}$ are continuous across a surface of discontinuity in material properties. In the domain outside the antenna system the scattered field $(\mathbf{E}^{\text{s}}, \mathbf{H}^{\text{s}})$ is introduced as the difference between the (actual) total field $(\mathbf{E}^{\text{R}}, \mathbf{H}^{\text{R}})$ and the incident field $(\mathbf{E}^{\text{i}}, \mathbf{H}^{\text{i}})$, the latter field being the electromagnetic field that would be present if the antenna system were absent:

$$\mathbf{E}^{\text{s}} \stackrel{\text{def}}{=} \mathbf{E}^{\text{R}} - \mathbf{E}^{\text{i}},\ \mathbf{H}^{\text{s}} \stackrel{\text{def}}{=} \mathbf{H}^{\text{R}} - \mathbf{H}^{\text{i}}. \tag{4.3}$$

The incident field is the plane wave:

$$\mathbf{E}^{\text{i}} = \mathbf{B} \exp(-ik_0\hat{\boldsymbol{\beta}} \cdot \mathbf{r}), \tag{4.4}$$

$$\mathbf{H}^{\text{i}} = (\varepsilon_0/\mu_0)^{\frac{1}{2}} (\mathbf{B} \times \hat{\boldsymbol{\beta}}) \exp(-ik_0\hat{\boldsymbol{\beta}} \cdot \mathbf{r}), \tag{4.5}$$

where $\mathbf{B}$ specifies the amplitude and the state of polarization (in general, elliptic) and $-\hat{\boldsymbol{\beta}}$ is the unit vector in the direction of propagation. Since the wave is transverse, we have $\mathbf{B} \cdot \hat{\boldsymbol{\beta}} = 0$. In the domain outside the antenna system both the incident and the scattered field satisfy the homogeneous Maxwell equations:

$$\operatorname{rot} \mathbf{H}^{\text{i,s}} + i\omega\varepsilon_0 \mathbf{E}^{\text{i,s}} = \mathbf{0}, \tag{4.6}$$

$$\operatorname{rot} \mathbf{E}^{\text{i,s}} - i\omega\mu_0 \mathbf{H}^{\text{i,s}} = \mathbf{0}. \tag{4.7}$$

In addition, the scattered field satisfies the radiation conditions (Cf. [3]).

5. The reciprocity relation

The reciprocity relation is obtained by an application of Lorentz's reciprocity theorem of electromagnetic fields [2]. This theorem can be applied, provided that the electromagnetic properties of the medium present in the transmitting situation and those of the medium present in the receiving situation are interrelated in such a way that at all points in space the relation:

$$\mathbf{E}^{\text{R}} \cdot \mathbf{D}^{\text{T}} - \mathbf{E}^{\text{T}} \cdot \mathbf{D}^{\text{R}} - \mathbf{H}^{\text{R}} \cdot \mathbf{B}^{\text{T}} + \mathbf{H}^{\text{T}} \cdot \mathbf{B}^{\text{R}} = 0 \tag{5.1}$$

holds. In the domain outside the antenna system this is obviously the case, as the constitutive equations are here simply $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$, both in the transmitting and in the receiving situation. In the antenna system the situation may be more complicated. Equation (5.1) holds *without change* of the properties of the medium if the medium is *reciprocal*. In all other cases the medium is non-reciprocal, and a change in properties has to be made when switching from transmission to reception, and vice versa. When, for example, unidirectional devices are present, the direction of blocking is to be reversed. It is noted that in the general theorem non-reciprocal media as well as media showing the magneto-electric effect [4] are included.

Employing the integral representation of $\mathbf{F}^{\mathrm{T}}$ indicated in Section 3, application of the reciprocity theorem to the domain outside the antenna system leads to:

$$\iint_S(\mathbf{E}^{\mathrm{T}} \times \mathbf{H}^{\mathrm{R}} - \mathbf{E}^{\mathrm{R}} \times \mathbf{H}^{\mathrm{T}}) \cdot \mathbf{n}\mathrm{d}A = (\mathrm{i}\omega\mu_0)^{-1}\mathbf{B} \cdot \mathbf{F}^{\mathrm{T}}(\hat{\boldsymbol{\beta}}), \quad (5.2)$$

where $\mathbf{n}$ denotes the unit vector in the direction of the outward normal to S. Application of the reciprocity theorem to the domain V occupied by the antenna system leads to:

$$\iint_S(\mathbf{E}^{\mathrm{T}} \times \mathbf{H}^{\mathrm{R}} - \mathbf{E}^{\mathrm{R}} \times \mathbf{H}^{\mathrm{T}}) \cdot \mathbf{n}\mathrm{d}A =$$

$$= \iiint_{V_{\text{source}}} (\mathbf{J}^{\mathrm{T}} \cdot \mathbf{E}^{\mathrm{R}} - \mathbf{K}^{\mathrm{T}} \cdot \mathbf{H}^{\mathrm{R}})\mathrm{d}V. \quad (5.3)$$

As the left-hand sides of (5.2) and (5.3) are equal, these equations can be combined to:

$$\iiint_{V_{\text{source}}} (\mathbf{J}^{\mathrm{T}} \cdot \mathbf{E}^{\mathrm{R}} - \mathbf{K}^{\mathrm{T}} \cdot \mathbf{H}^{\mathrm{R}})\mathrm{d}V = (\mathrm{i}\omega\mu_0)^{-1}\mathbf{B} \cdot \mathbf{F}^{\mathrm{T}}(\hat{\boldsymbol{\beta}}). \quad (5.4)$$

Details of the proofs are given in [2].

The reciprocity relation for a particular antenna system at hand can be obtained from either (5.2), (5.3) or (5.4). Which one of these equations is to be selected as a starting point, depends mainly on where the information concerning the antenna properties is needed. In the next section this will be illustrated for a microwave antenna system where the field quantities are measured in a uniform section of the cylindrical waveguide feeding the antenna system (see also [5]).

6. Application to a microwave antenna system

In this section we discuss the application of the reciprocity theorem of Section 5 to the microwave antenna system depicted in Fig. 1. In the transmitting situation the domain indicated as the generator is capable of carrying external currents, in the receiving situation this part of the system acts as the load. The waveguide connecting the generator/load domain to the other parts of the antenna system (amongst which are the reflector or horn and the radiating aperture) is assumed to have a uniformly cylindrical, lossless section. Let M be the (finite) number of propagating modes in this section; all modes of higher order than M are evanescent.

To allow for multimode operation*) of the waveguide, we take $M \geqq 1$. It is further assumed that in the uniform section a transverse reference plane can be chosen, such that with sufficient accuracy the total electromagnetic field in this cross-section can be written as the superposition of the contributions from the propagating modes only. (This implies that the reference plane is sufficiently far removed from non-uniformities in the waveguide.) Let the Cartesain co-ordinate system be chosen such that the plane $z = L$ coincides with the reference plane and the positive z-direction be chosen away from the generator/load. Then the transverse parts of the electromagnetic field vectors in the reference plane can be written as:

$$\mathbf{E}_{\text{transverse}}(x,y,L) =$$

$$= \Sigma_{m=1}^{M}[A_m \exp(\mathrm{i}\beta_m L) + B_m \exp(-\mathrm{i}\beta_m L)]\mathbf{e}_m(x,y), \quad (6.1)$$

$$\mathbf{H}_{\text{transverse}}(x,y,L) =$$

$$= \Sigma_{m=1}^{M}[A_m \exp(\mathrm{i}\beta_m L) - B_m \exp(-\mathrm{i}\beta_m L)]\mathbf{h}_m(x,y), \quad (6.2)$$

where A_m is the complex amplitude of the m-th mode propagating in the positive z-direction, B_m is the complex amplitude of the m-th mode in the negative z-direction and β_m, $\mathbf{e}_m$ and $\mathbf{h}_m$ are the phase factor, the transverse electric field distribution and the transverse magnetic field distribution, respectively, of the m-th mode propagating in the positive z-direction. The transverse mode distributions satisfy the orthogonality relation:

$$\iint_D(\mathbf{e}_m \times \mathbf{h}_n) \cdot \mathbf{i}_z\mathrm{d}x\mathrm{d}y = 0 \text{ if } m \neq n, \quad (6.3)$$

where D is the cross-section of the waveguide and $\mathbf{i}_z$ is the unit vector in the positive z-direction. Further, the normalization constant N_m of the m-th mode is introduced as:

$$N_m \stackrel{\text{def}}{=} \iint_D(\mathbf{e}_m \times \mathbf{h}_m) \cdot \mathbf{i}_z\mathrm{d}x\mathrm{d}y. \quad (6.4)$$

In the *transmitting situation* we write:

$$B_m^{\mathrm{T}} = \rho_m^{\mathrm{T}} A_m^{\mathrm{T}}, \quad (6.5)$$

*) The question as to multimode operation has been raised by Dr. M. E. J. Jeuken, Department of Electrical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands.

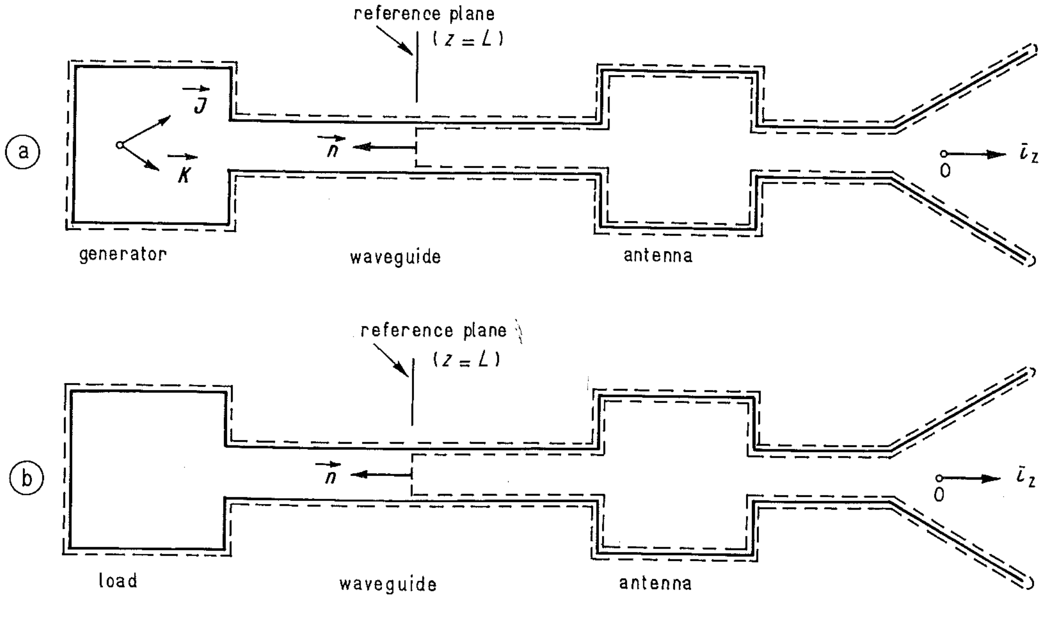

Fig. 1. Reciprocity relation for a microwave antenna system. In the transmitting situation (a) the generator domain is capable of carrying external currents, in the receiving situation (b) a plane wave is incident upon the system.

where ρ^{T}_{m} is the reflection factor of the m-th mode in the transmitting situation. This factor is a measure for the mismatch of the radiating part of the antenna with respect to the m-th waveguide mode; if the relevant part of the antenna system is matched to the m-th waveguide mode we have $\rho^{T}_{m} = 0$. Due to the linearity of the system we can further write:

$$\mathbf{F}^{T}(\hat{\mathbf{r}}_{P}) = \Sigma^{M}_{m=1} A^{T}_{m} \mathbf{F}^{T}_{m}(\hat{\mathbf{r}}_{P}), \tag{6.6}$$

where $\mathbf{F}^{T}_{m}(\hat{\mathbf{r}}_{P})$ is the contribution from the m-th mode of unit amplitude to the vectorial amplitude radiation characteristic of the antenna in the transmitting situation.
In the *receiving situation* we write:

$$A^{R}_{m} = \rho^{R}_{m} B^{R}_{m}, \tag{6.7}$$

where ρ^{R}_{m} is the reflection factor of the m-th mode in the receiving situation. This factor is a measure for the mismatch of the load with respect to the m-th waveguide mode; if the load is matched to the m-th waveguide mode we have $\rho^{R}_{m} = 0$.

We now apply (5.2) to the closed surface S indicated by the dotted lines in Fig. 1. Since the waveguide walls, all other walls of the antenna system and the reflector or horn are assumed to be electrically perfectly conducting, we have $\mathbf{n} \times \mathbf{E}^{T,R} = \mathbf{0}$ on these parts. Hence:

$$\int\int_{S}(\mathbf{E}^{T} \times \mathbf{H}^{R} - \mathbf{E}^{R} \times \mathbf{H}^{T}) \cdot \mathbf{n} \mathrm{d}A =$$
$$= -\int\int_{D}(\mathbf{E}^{T} \times \mathbf{H}^{R} - \mathbf{E}^{R} \times \mathbf{H}^{T}) \cdot \mathbf{i}_{z} \mathrm{d}x\mathrm{d}y. \tag{6.8}$$

Substituting the expansions (6.1) and (6.2) in the right-hand side, taking into account the orthogonality relation (6.3) and the normalization condition (6.4) and using (6.5), (6.6) and (6.7), we obtain:

$$2\Sigma^{M}_{m=1}(1-\rho^{T}_{m}\rho^{R}_{m}) A^{T}_{m} B^{R}_{m} N_{m} = (\mathrm{i}\omega\mu_{0})^{-1} \mathbf{B} \cdot \Sigma^{M}_{m=1} A^{T}_{m} \mathbf{F}^{T}_{m}(\hat{\boldsymbol{\beta}}). \tag{6.9}$$

As (6.9) must hold for arbitrary values of $A^{T}_{1}, \ldots, A^{T}_{M}$ we have:

$$2(1-\rho^{T}_{m}\rho^{R}_{m}) B^{R}_{m} N_{m} = (\mathrm{i}\omega\mu_{0})^{-1} \mathbf{B} \cdot \mathbf{F}^{T}_{m}(\hat{\boldsymbol{\beta}}) \qquad (m = 1, \ldots, M). \tag{6.10}$$

By measuring ρ^{T}_{m}, ρ^{R}_{m} and B^{R}_{m} and calculating N_{m} we can determine $\mathbf{B} \cdot \mathbf{F}^{T}_{m}(\hat{\boldsymbol{\beta}})$ from (6.10). By varying the direction of propagation $\hat{\boldsymbol{\beta}}$ of the incident plane wave and its state of polarization we then can determine all components of $\mathbf{F}^{T}_{m}(\hat{\boldsymbol{\beta}})$ for all relevant directions (Cf. [5]).

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