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THE VECTOR INTEGRAL-EQUATION METHOD FOR COMPUTING THREE-DIMENSIONAL MAGNETIC FIELDS

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Abstract

The vector integral-equation method for computing three-dimensional, quasi-static magnetic fields is developed with a view to its application to configurations of the type that occur in magnetic recording. Starting from appropriate Green-type vector integral relations for the magnetic-field quantities, the relevant integral equations are derived. Their numerical handling is briefly discussed.

Contents

1. INTRODUCTION

Integral-equation methods have proved their usefulness for calculating electromagnetic fields in a variety of configurations. Especially, they have been used extensively in analyzing electromagnetic scattering and diffraction problems (cf. De Hoop 1977). The main advantage of the method lies in its flexibility as regards shape, size and physical properties of the different consituents that together form the configuration. Its implementation on the computer offers no extreme difficulties and in this respect, the only limitations seem to be the computation time on and the storage capacity of the computer system at one's disposal.

For the computation of quasi-static magnetic fields, domain-type as well as boundary-type integral-equation methods have been developed (Holzinger 1970, Banchev and Voroszhtsov 1976, Iselin 1976, Simkin and Trowbridge 1976, Trowbridge 1976). In many cases (Simkin and Trowbridge 1978, Armstrong et al. 1978, Carpenter 1977), either magnetic scalar or magnetic vector potentials are introduced to arrive at the desired source representations that are needed as a starting point for the integral-equation formulation. However, the potentials suffer from the disadvantage that magnetic-field boundary conditions are not directly expressed in terms of them. For this reason, expressions in which only the magnetic-field quantities themselves occur, are to be preferred. The starting point for deriving such a relation is the vector Green identity of the third kind. Application of this identity to the magnetic-field strength or to the magnetic-flux density leads, after some transformations of the relevant integrals, to integral relations that hold for piecewise continuous source distributions.

In the configuration that we are going to discuss, three different kinds of objects are present in otherwise free space: (1) objects of infinite permeability, (2) perfectly conducting objects, (3) non-conducting objects with a finite permeability. In objects of the kinds (1) and (2), the magnetic field cannot penetrate; as a consequence, their influence on the magnetic field is accounted for by boundary conditions invoked at their boundary surfaces. In objects of the kind (3), the magnetic field does penetrate and their influence on the magnetic field is accounted for by invoking, in their interior, the constitutive relation pertinent to the magnetic material of which they consist. The configuration is excited by a current carrying coil with prescribed distribution of the volume current density (Fig. 1).

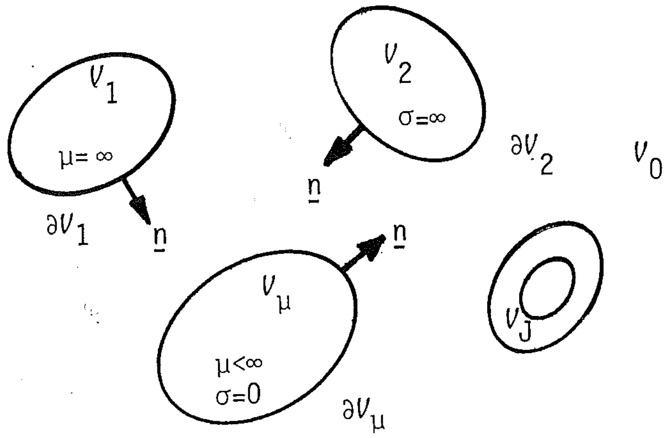

Fig. 1. Configuration for which the integral-equation formulation of magnetic-field problems is developed.

The dimensions of the configuration are assumed to be so small, that the travel time for electromagnetic waves to traverse it, is negligible on the time scale on which the magnetic field varies. Then, the quasi-static approximation of magnetic-field theory applies.

2. DESCRIPTION OF THE CONFIGURATION

In the configuration, three different kinds of objects are present in otherwise free space. Further, a current carrying coil that excites the configuration, is present. The nomenclature used to distinguish the different subdomains, is given in Table 1. It is assumed that the intersection of each pair of domains out of V_μ, V_1, V_2 and V_J is empty (Fig. 1).

Table 1. Nomenclature adopted for the different subdomains of the configuration

domain	boundary surface	physical property
V_μ	∂V_μ	non-conducting material of finite permeability
V_1	∂V_1	non-conducting material of infinite permeability
V_2	∂V_2	perfectly conducting material
V_0		vacuum domain
V_J		non-magnetic material carrying prescribed external current

To locate a point in the configuration, we employ orthogonal, Cartesian coordinates x,y,z with respect to a given, orthogonal, Cartesian reference frame that is at rest with respect to the (stationary) objects. The reference frame is specified by its origin O and the three mutually perpendicular base vectors of unit length $\underline{i}_x$, $\underline{i}_y$, $\underline{i}_z$. In the given order, the base vectors form a right-handed system. The position vector is denoted by $\underline{r}$; it is given by

$$\underline{r} = x\underline{i}_x + y\underline{i}_y + z\underline{i}_z. \tag{1}$$

The time coordinate is denoted by t.

3. THE MAGNETIC FIELD IN THE CONFIGURATION

We employ the quasi-static approximation of the magnetic-field equations. Then, the magnetic state in the configuration involves the following quantities:

$\underline{H}$ = magnetic-field strength (A/m),
$\underline{B}$ = magnetic-flux density (T),
$\underline{M}$ = magnetization (A/m),
$\underline{J}$ = volume density of electric current (A/m^2),

while the induced electric field is

$\underline{E}$ = electric-field strength (V/m).

In parantheses, the pertaining SI-units have been indicated.

Field equations

At any interior point of a domain where the magnetic properties either do not vary or vary continuously with position, the field quantities are continuously differentiable and satisfy the field equations

$$\underline{\nabla} \times \underline{H} = \underline{J}, \tag{2}$$

$$\underline{\nabla} \times \underline{E} = - \partial_t \underline{B}, \tag{3}$$

$$\underline{\nabla} \cdot \underline{B} = 0, \tag{4}$$

$$\underline{\nabla} \cdot \underline{J} = 0, \tag{5}$$

while

$$\underline{B} = \mu_0(\underline{H} + \underline{M}), \tag{6}$$

where

$$\mu_0 = 4\pi \cdot 10^{-7}\ \text{H/m} \tag{7}$$

is the permeability in vacuo. The operator $\underline{\nabla}$ is given by

$$\underline{\nabla} = \underline{i}_x\partial_x + \underline{i}_y\partial_y + \underline{i}_z\partial_z \tag{8}$$

and partial differentiation is denoted by ∂.

Constitutive relations

First of all, we assume that no permanent magnetization is present. Hence $\underline{M}$ only contains a field-dependent part and the interrelation between $\underline{M}$ and $\underline{H}$ is taken as constitutive relation. If now, the magnetic materials present in the configuration are instantaneously reacting and time invariant, the quantities $\underline{H}$, $\underline{B}$ and $\underline{M}$ all have the same time dependence as the exciting current with volume density $\underline{J}$. In this case, the calculation of the magnetic field in the configuration is a problem in space only. In all other cases, the calculation of the field is a problem in space-time. In particular, the latter applies to all cases where the magnetic behaviour of the materials is non-linear and/or hysteretic. One of the simplest cases arises if the material is linear, time invariant, instantaneously and locally reacting, and isotropic. Then, we have

$$\underline{M}(\underline{r},t) = \chi_m(\underline{r})\ \underline{H}(\underline{r},t), \tag{9}$$

where χ_m is the magnetic susceptibility of the medium.

Boundary conditions

At surfaces across which the properties of the media show abrupt changes, the field equations (2)-(5) have to be supplemented by boundary conditions. These are

$$\underline{n} \times \underline{H} \text{ and } \underline{n} \cdot \underline{B} \text{ continuous across } \partial V_{\mu}, \tag{10}$$

$$\underline{n} \times \underline{H} \to \underline{0} \quad \text{upon approaching } \partial V_1, \tag{11}$$

$$\underline{n} \cdot \underline{B} \to 0 \quad \text{upon approaching } \partial V_2. \tag{12}$$

Here, $\underline{n}$ denotes the unit vector along the normal to the relevant boundary surface.

Conditions at infinity

For any configuration of the type under consideration where the non-vacuum parts occupy a bounded subdomain of $\mathbf{R}^3$, we have

$$\{\underline{H}, \underline{B}\} = \text{Order}(|\underline{r}|^{-3}) \quad \text{as } |\underline{r}| \to \infty, \text{ uniformly in all directions.} \tag{13}$$

4. GREEN-TYPE INTEGRAL RELATIONS FOR THE MAGNETIC-FIELD STRENGTH AND THE MAGNETIC-FLUX DENSITY

The starting point for the vector integral-equation formulation of magnetic-field problems in the configuration described in Section 2 are Green-type integral relations for the magnetic-field strength and the magnetic-flux density. These follow from an application of the vector Green identity of the third kind (A.10), in which $\underline{Q}$ is successively identified with $\underline{H}$ or $\underline{B}$.

Green-type integral relation for the magnetic-field strength

In (A.10), we identify $\underline{Q}$ with $\underline{H}$. Recalling that

$$\underline{\nabla} \cdot \underline{H} = - \underline{\nabla} \cdot \underline{M}, \tag{14}$$

$$\underline{\nabla} \times \underline{H} = \underline{J}, \tag{15}$$

and hence,

$$(\underline{\nabla} \cdot \underline{\nabla})\underline{H} = - \underline{\nabla} \times \underline{J} - \underline{\nabla}(\underline{\nabla} \cdot \underline{M}), \tag{16}$$

we obtain (cf. Fig. A)

$$\int_{\partial V} [- (\underline{\nabla} \cdot \underline{M})\underline{n}G + \underline{J} \times \underline{n}G - (\underline{n} \cdot \underline{H})\underline{\nabla}G - (\underline{n} \times \underline{H}) \times \underline{\nabla}G]dA - \int_V G[- \underline{\nabla} \times \underline{J}$$

$$- \underline{\nabla}(\underline{\nabla} \cdot \underline{M})]dV = \{1,\tfrac{1}{2},0\}\underline{H}(\underline{r}') \quad \text{if } \underline{r}' \in \{V,\partial V,V'\}. \tag{17}$$

Now, (17) only holds if the right-hand side of (16) is piecewise continuous, which is a rather severe restriction. In view of the application of the theory to the computation of magnetic fields, we would like to have a relation of the type (17), but for piecewise continuous distributions of $\underline{J}$ and $\underline{M}$. Such a relation is arrived at by using the property

$$\underline{\nabla}G = - \underline{\nabla}'G, \tag{18}$$

where

$$\underline{\nabla}' = \underline{i}_x\partial_{x'} + \underline{i}_y\partial_{y'} + \underline{i}_z\partial_{z'}, \tag{19}$$

and applying appropriate theorems of vector analysis to the left-hand side of (17). In this respect, the following steps are carried out:

$$\int_V G(\underline{\nabla} \times \underline{J})dV = \int_V [\underline{\nabla} \times (G\underline{J}) - (\underline{\nabla}G) \times \underline{J}]dV$$

$$= \int_{\partial V} \underline{n} \times G\underline{J}\, dA + \underline{\nabla}' \times \int_V G\underline{J}\, dV, \tag{20}$$

$$\int_V G\, \underline{\nabla}(\underline{\nabla} \cdot \underline{M})dV = \int_V \{\underline{\nabla}[G(\underline{\nabla} \cdot \underline{M})] - (\underline{\nabla}G)(\underline{\nabla} \cdot \underline{M})\}dV$$

$$= \int_{\partial V} (\underline{\nabla} \cdot \underline{M})\underline{n}G\, dA + \underline{\nabla}' \int_V G(\underline{\nabla} \cdot \underline{M})dV \tag{21}$$

and

$$\int_V G(\underline{\nabla} \cdot \underline{M})dV = \int_V [\underline{\nabla} \cdot (G\underline{M}) - (\underline{\nabla}G) \cdot \underline{M}]dV$$

$$= \int_{\partial V} G(\underline{n} \cdot \underline{M})dA + \underline{\nabla}' \cdot \int_V G\underline{M}\, dV. \quad (22)$$

Using (18)-(22) in (17), we obtain

$$\underline{\nabla}' \int_{\partial V} G(\underline{n} \cdot \underline{B}/\mu_0)dA - \underline{\nabla}' \times \int_{\partial V} G(\underline{n} \times \underline{H})dA$$

$$+ \underline{\nabla}' \times \int_V G\underline{J}\, dV + \underline{\nabla}'(\underline{\nabla}' \cdot \int_V G\underline{M}\, dV) = \{1,\tfrac{1}{2},0\}\underline{H}(\underline{r}') \quad \text{if } \underline{r}' \in \{V,\partial V,V'\}.$$

$$(23)$$

Equation (23) is the desired Green-type integral relation for the magnetic-field strength. Note, that the integrands in the boundary integrals are continuous upon crossing an interface across which $\underline{J}$ and/or $\underline{M}$ jump by finite amounts. Hence, these interfaces do not contribute to the left-hand side if (23) is applied to adjacent domains and the resulting equations are added. In the final result, only boundary integrals over the boundary of the domain of application remain and (23) holds for piecewise continuous distributions of $\underline{J}$ and $\underline{M}$. For points of observation on an interface, the value of $\underline{H}(\underline{r}')$ is to be replaced by half the sum of the limiting values on either side of the interface.

Green-type integral relation for the magnetic-flux density

In (A.10), we identify $\underline{Q}$ with $\underline{B}$. Recalling that

$$\underline{\nabla} \cdot \underline{B} = 0, \quad (24)$$

$$\underline{\nabla} \times \underline{B} = \mu_0\underline{J} + \mu_0\underline{\nabla} \times \underline{M}, \quad (25)$$

and hence,

$$(\underline{\nabla} \cdot \underline{\nabla})\underline{B} = -\mu_0(\underline{\nabla} \times \underline{J}) - \mu_0\underline{\nabla} \times (\underline{\nabla} \times \underline{M}), \quad (26)$$

we obtain (cf. Fig. A)

$$\int_{\partial V} [(\mu_0\underline{J} + \mu_0\underline{\nabla} \times \underline{M}) \times \underline{n}G - (\underline{n} \cdot \underline{B})\underline{\nabla}G - (\underline{n} \times \underline{B}) \times \underline{\nabla}G]dA - \int_V G[-\mu_0\underline{\nabla}\times \underline{J}$$

$$- \mu_0\underline{\nabla} \times (\underline{\nabla} \times \underline{M})]dV = \{1,\tfrac{1}{2},0\}\underline{B}(\underline{r}') \quad \text{if } \underline{r}' \in \{V,\partial V,V'\}. \tag{27}$$

Equation (27) only holds if the right-hand side of (26) is piecewise continuous. A relation of a similar kind, but for only piecewise continuous distributions of $\underline{J}$ and $\underline{M}$, is arrived at in a similar way as above. First of all, (18) -(20) are used; next we employ the relations

$$\int_V G\,[\underline{\nabla} \times (\underline{\nabla} \times \underline{M})]dV = \int_V \{\underline{\nabla} \times [G(\underline{\nabla} \times \underline{M})] - (\underline{\nabla}G) \times (\underline{\nabla} \times \underline{M})\}dV$$

$$= \int_{\partial V} \underline{n} \times G(\underline{\nabla} \times \underline{M})dA + \underline{\nabla}' \times \int_V G(\underline{\nabla} \times \underline{M})dV \tag{28}$$

and

$$\int_V G(\underline{\nabla} \times \underline{M})dV = \int_V [\underline{\nabla} \times (G\underline{M}) - (\underline{\nabla}G) \times \underline{M}]dV$$

$$= \int_{\partial V} \underline{n} \times (G\underline{M})dA + \underline{\nabla}' \times \int_V G\underline{M}\, dV. \tag{29}$$

Using (18)-(20), (28) and (29) in (27), we obtain

$$\underline{\nabla}' \int_{\partial V} G(\underline{n} \cdot \underline{B})dA - \underline{\nabla}' \times \int_{\partial V} G(\underline{n} \times \mu_0\underline{H})dA + \underline{\nabla}' \times \int_V G\mu_0\underline{J}\, dV$$

$$+ \underline{\nabla}' \times (\underline{\nabla}' \times \int_V G\mu_0\underline{M}\, dV) = \{1,\tfrac{1}{2},0\}\underline{B}(\underline{r}') \quad \text{if } \underline{r}' \in \{V,\partial V,V'\}. \tag{30}$$

Equation (30) is the desired Green-type integral relation for the magnetic-flux density. Note, that the integrands in the boundary integrals are continuous upon crossing an interface across which $\underline{J}$ and/or $\underline{M}$ jump by finite amounts. Hence, these interfaces do not contribute to the left-hand side if (30) is applied to adjacent domains and the resulting equations are added. In the final result, only boundary integrals over the boundary of the domain of application remain and (30) holds for piecewise continuous distributions of $\underline{J}$ and $\underline{M}$. For points of observation on an interface, the value of $\underline{B}(\underline{r}')$ is to be replaced by half the sum of the limiting values on either side of the interface.

The integral relations (23) and (30) have been derived for the bounded domain interior to the bounded closed surface ∂V. They equally well apply to the unbounded domain exterior to a bounded closed surface, provided that the conditions at infinity (13) are invoked. Note, that for (23) and (30) to hold in this case, the unit vector $\underline{n}$ along the normal to ∂V must again point away from V. As far as jumps in $\underline{J}$ are concerned, these are, on account of (5), admissible in its tangential components only.

5. INTEGRAL-EQUATION FORMULATION OF THE MAGNETIC-FIELD PROBLEM

The integral relations (23) and (30) are now employed to arrive at the integral-equation formulation of the magnetic-field problem pertaining to the configuration described in Section 2. Taking into account the boundary conditions on ∂V_1 (cf. (11)) and ∂V_2 (cf. (12)), we introduce the following quantities

$$\Phi_1(\underline{r}') = \int_{\partial V_1} G(\underline{n} \cdot \underline{B}/\mu_0)dA, \tag{31}$$

$$\underline{A}_2(\underline{r}') = \int_{\partial V_2} G(\underline{n} \times \mu_0\underline{H})dA, \tag{32}$$

$$\underline{A}_J(\underline{r}') = \int_{V_J} G\mu_0\underline{J}\, dV, \tag{33}$$

$$\underline{\Pi}_\mu(\underline{r}') = \int_{V_\mu} G\underline{M}\, dV. \tag{34}$$

Using the integral relation (23) for the magnetic-field strength and recalling that on ∂V_1 and ∂V_2 the unit vector $\underline{n}$ along the normal is taken as in Fig. 1 (i.e. pointing toward V_0), it follows that

$$- \underline{\nabla}'\Phi_1 + \mu_0^{-1}\underline{\nabla}' \times \underline{A}_2 + \mu_0^{-1}\underline{\nabla} \times \underline{A}_J$$

$$+ \underline{\nabla}'(\underline{\nabla}' \cdot \underline{\Pi}_\mu) = \{1,\tfrac{1}{2},0\}\underline{H}(\underline{r}') \quad \text{if } \underline{r}' \in \{V_0, \partial V_0, V_0'\}. \tag{35}$$

Similarly, the integral relation (30) for the magnetic-flux density yields

$$- \mu_0 \underline{\nabla}' \Phi_1 + \underline{\nabla}' \times \underline{A}_2 + \underline{\nabla}' \times \underline{A}_J$$

$$+ \mu_0 \underline{\nabla}' \times (\underline{\nabla}' \times \underline{\Pi}_\mu) = \{1, \tfrac{1}{2}, 0\} \underline{B}(\underline{r}') \quad \text{if } \underline{r}' \in \{V_0, \partial V_0, V_0'\}. \tag{36}$$

In these expressions, $\underline{n} \cdot \underline{B}/\mu_0$ is an unknown scalar quantity on ∂V_1, while $\underline{n} \times \underline{H}$ is an unknown, two-component, vector quantity on ∂V_2. In V_J, $\underline{J}$ is assumed to be known, while in V_μ , the quantities $\underline{M}$ and $\underline{H}$ are interrelated through the constitutive relation, and either of the two can be considered as unknown.

By taking, in (35) and (36), $\underline{r}' \in \partial V_1$, $\underline{r}' \in \partial V_2$ and $\underline{r}' \in V_\mu$, respectively, a number of integral equations is obtained from which the unknown field distributions can, in principle, be determined. Some of the integral equations are of the first kind, others are of the second kind (cf. Table 2). In the next section, some features of these integral equations will be discussed.

Table 2. Integral equations obtained from the Green-type integral relations

domain	expression for the field quantity	unknown field quantity	resulting integral equation
∂V_1	$\underline{n} \cdot \underline{B}/\mu_0$	$\underline{n} \cdot \underline{B}/\mu_0$	2nd kind
∂V_1	$\underline{n} \times \underline{H}$	$\underline{n} \cdot \underline{B}/\mu_0$	1st kind
∂V_2	$\underline{n} \cdot \underline{B}/\mu_0$	$\underline{n} \times \underline{H}$	1st kind
∂V_2	$\underline{n} \times \underline{H}$	$\underline{n} \times \underline{H}$	2nd kind
V_μ	$\underline{H}$	$\underline{H}$ or $\underline{M}$	2nd kind

6. THE MAGNETIC-FIELD INTEGRAL EQUATIONS AND SOME OF THEIR PROPERTIES

For all configurations met in practice, the integral equations referred to in Table 2 have to be solved numerically. This implies that, in three different respects, approximations are made:

(1) the unknown function is approximated by a finite sequence of "expansion functions" the coefficients of which are to be determined;

(2) the resulting integrals are, in most cases, evaluated numerically (i.e. approximately) by a suitable numerical integration formula;

(3) the equality sign in the equations is satisfied in some approximate sense, usually by weighting the integral equations through an appropriate sequence of "weighting functions" over their domain of application (the collocation method can be considered as a special case of this).

The question now arises, how the relevant approximations made on ∂V_1 and ∂V_2 and in V_μ manifest themselves in V_0. This is the more important in those cases where the computed field in V_0 is used for further processing to yield certain characteristics of the magnetic system as a whole. As an example, we mention magnetic-recording configurations, where the computed external field is used to compute the harmonic or the digital response of the entire recording system. In such applications, it is reasonable to require the field to satisfy, in V_0, the relevant magnetic-field equations (cf. Section 3). Now, this is in general not the case, unless the approximate surface distributions satisfy the compatibility relations

$$\int_C \underline{\tau} \cdot \underline{H}\, ds = 0, \tag{37}$$

for any closed curve C, with unit tangent vector $\underline{\tau}$, that is situated on either ∂V_1 or ∂V_2, and

$$\int_{\partial V_1} \underline{n} \cdot \underline{B}\, dA = 0. \tag{38}$$

Equation (37) follows from (2), together with the condition that no electric current flows into or out of either ∂V_1 or ∂V_2. Equation (38) follows from

(4), together with the boundary condition on ∂V_2 and the condition at infinity (13), and guarantees that, at infinity, the magnetic field is of a dipole character.

Equation (37) makes also clear, that for the single unknown quantity on ∂V_1 we can either select a single integral equation of the second kind or a two-component vector integral equation of the first kind, while on ∂V_2 the two-component unknown vectorial quantity itself must satisfy (37). For the way in which (36) and (37) can be implemented, as well as for the application of the theory to a number of magnetic-recording configurations we refer to Van Herk (1980).

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APPENDIX

A. VECTOR GREEN IDENTITIES

By analogy with potential theory, three different kinds of vector identities of the Green type will be distinguished. The identities of the first and the second kind apply to two vector functions of position, $\underline{P} = \underline{P}(\underline{r})$ and $\underline{Q} = \underline{Q}(\underline{r})$, that are defined in a bounded domain V and on its boundary surface ∂V, and are continuously differentiable in V and on ∂V the required number of times. In the identity of the third kind, we choose $\underline{P} = \underline{a}\, G$, where $\underline{a}$ is an arbitrary constant vector and G is a scalar Green's function that is singular at $\underline{r} = \underline{r}'$. In this latter case, three positions of $\underline{r}'$ have to be distinguished, viz. $\underline{r}' \in V$, $\underline{r}' \in \partial V$ and $\underline{r}' \in V'$, where V' is the complement of $V \cup \partial V$ in $\mathbb{R}^3$.

Vector Green identities of the first kind

Application of Gauss' divergence theorem to the vector function $\underline{P}(\underline{\nabla} \cdot \underline{Q})$ yields

$$\int_{\partial V} \underline{n} \cdot \underline{P}(\underline{\nabla} \cdot \underline{Q})\, dA = \int_V \{(\underline{\nabla} \cdot \underline{P})(\underline{\nabla} \cdot \underline{Q}) + \underline{P} \cdot [\underline{\nabla}(\underline{\nabla} \cdot \underline{Q})]\} dV. \qquad (A.1)$$

Application of Gauss' divergence theorem to the vector function $\underline{P} \times (\underline{\nabla} \times \underline{Q})$ yields

$$\int_{\partial V} \underline{n} \cdot [\underline{P} \times (\underline{\nabla} \times \underline{Q})] dA = \int_V \{(\underline{\nabla} \times \underline{P}) \cdot (\underline{\nabla} \times \underline{Q}) - \underline{P} \cdot [\underline{\nabla} \times (\underline{\nabla} \times \underline{Q})]\} dV. \qquad (A.2)$$

Equations (A.1) and (A.2) are vector Green identities of the first kind.

Vector Green identities of the second kind

From (A.1) and a similar formula that results from interchanging $\underline{P}$ and $\underline{Q}$, we obtain

$$\int_{\partial V} \underline{n} \cdot [\underline{P}(\underline{\nabla} \cdot \underline{Q}) - \underline{Q}(\underline{\nabla} \cdot \underline{P})] dA = \int_V \{\underline{P} \cdot [\underline{\nabla}(\underline{\nabla} \cdot \underline{Q})] - \underline{Q} \cdot [\underline{\nabla}(\underline{\nabla} \cdot \underline{P})]\} dV. \qquad (A.3)$$

From (A.2) and a similar formula that results from interchanging $\underline{P}$ and $\underline{Q}$, we obtain

$$\int_{\partial V} \underline{n} \cdot [\underline{P} \times (\underline{\nabla} \times \underline{Q}) - \underline{Q} \times (\underline{\nabla} \times \underline{P})]dA$$

$$= \int_V \{- \underline{P} \cdot [\underline{\nabla} \times (\underline{\nabla} \times \underline{Q})] + \underline{Q} \cdot [\underline{\nabla} \times (\underline{\nabla} \times \underline{P})]\}dV. \quad \text{(A.4)}$$

Addition of (A.3) and (A.4) yields

$$\int_{\partial V} \underline{n} \cdot [\underline{P}(\underline{\nabla} \cdot \underline{Q}) + \underline{P} \times (\underline{\nabla} \times \underline{Q}) - \underline{Q}(\underline{\nabla} \cdot \underline{P}) - \underline{Q} \times (\underline{\nabla} \times \underline{P})]dA$$

$$= \int_V \{\underline{P} \cdot [(\underline{\nabla} \cdot \underline{\nabla})\underline{Q}] - \underline{Q} \cdot [(\underline{\nabla} \cdot \underline{\nabla})\underline{P}]\}dV, \quad \text{(A.5)}$$

where we have used the property

$$\underline{\nabla}(\underline{\nabla} \cdot \quad) - \underline{\nabla} \times (\underline{\nabla} \times \quad) = (\underline{\nabla} \cdot \underline{\nabla}) \quad . \quad \text{(A.6)}$$

Equations (A.3), (A.4) and (A.5) are vector Green identities of the second kind.

Vector Green identity of the third kind

The vector Green identity of the third kind is arrived at by taking, in (A.5), for the vector function $\underline{P}$ the expression

$$\underline{P} = \underline{a}\, G, \quad \text{(A.7)}$$

where $\underline{a}$ is an arbitrary constant vector, while $G = G(\underline{r}' - \underline{r})$ is given by

$$G = 1/4\pi|\underline{r}' - \underline{r}| \quad \text{with } \underline{r} \in \mathbb{R}^3, \underline{r}' \in \mathbb{R}^3. \quad \text{(A.8)}$$

In (A.8),

$$|\underline{r}' - \underline{r}| = [(\underline{r}' - \underline{r})\cdot(\underline{r}' - \underline{r})]^{\frac{1}{2}} \quad \text{with } [\ldots]^{\frac{1}{2}} \geq 0. \tag{A.9}$$

In these expressions, $\underline{r}'$ denotes the position vector of a fixed point of observation. Three positions of $\underline{r}'$ have to be distinguished, viz. $\underline{r}' \in V$, $\underline{r}' \in \partial V$ and $\underline{r}' \in V'$. If $\underline{r}' \in V$ or $\underline{r}' \in \partial V$, $\underline{P}$ is singular at $\underline{r} = \underline{r}'$. The usual procedure of 'excluding a sphere with vanishingly small radius' can be used to assign a meaning to (A.5) in these cases. In the resulting expressions, the factor $\underline{a}$ can, through the application of a number of vector formulas, be brought in front of the integral signs. The observation that $\underline{a}$ is arbitrary, then leads to

$$\int_{\partial V} [(\underline{\nabla}\cdot\underline{Q})\underline{n}G + (\underline{\nabla}\times\underline{Q})\times\underline{n}G - (\underline{n}\cdot\underline{Q})\underline{\nabla}G - (\underline{n}\times\underline{Q})\times\underline{\nabla}G]dA$$

$$- \int_V G[(\underline{\nabla}\cdot\underline{\nabla})\underline{Q}]dV = \{1,\tfrac{1}{2},0\}\underline{Q}(\underline{r}') \quad \text{if } \underline{r}' \in \{V, \partial V, V'\}. \tag{A.10}$$

Equation (A.10) is the vector Green identity of the third kind. The result for $\underline{r}' \in \partial V$ holds at points where ∂V is locally flat (Fig. A).

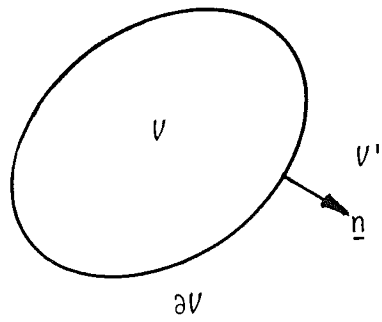

Fig. A. Configuration for which the Green identities are derived.

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