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MODERN TOPICS IN ELECTROMAGNETICS AND ANTENNAS

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6. GENERAL CONSIDERATIONS ON THE INTEGRAL-EQUATION FORMULATION OF DIFFRACTION PROBLEMS

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This chapter deals with the integral-equation formulation of electromagnetic scattering and diffraction problems. As compared with other techniques for calculating the electromagnetic field in the presence of a scattering object, the integral-equation method has the advantage of yielding accurate results for scattering objects whose shape, dimensions and physical properties can vary in a wide range. For any non-trivial problem, the method has to be implemented on a computer. The computational aspects of the method have led to our preference for an analysis that follows as closely as possible the different steps that the computer programme will have to carry out. In this respect our presentation differs somewhat from the ones that emphasize the operator formalism that underlies the method. Although the operator formalism is of vital importance to the understanding of the mathematics behind the use of integral equations, it is believed that an analysis close to the electromagnetics of the scattering problem has its own merits. From this latter point of view the present contribution has been written.

The first step in the integral-equation method consists of acquiring proper integral representations for the electromagnetic-field quantities involved. This is done in a rather unorthodox way by consistently using the spatial Fourier transform of the electromagnetic field equations pertaining to field quantities whose domain of definition is a subdomain of three-dimensional space (Section 6.5). Next, the integral representations are used to represent the scattered field (Section 6.6).

When this has been achieved, we are ready to obtain the desired integral equations. In this respect we distinguish between four different types of scattering objects, viz. (a) inhomogeneous, penetrable objects, (b) electrically impenetrable (i.e. perfectly conducting) objects, (c) homogeneous, penetrable objects, (d) objects of vanishing thickness. From the relevant integral equations (discussed in Sections 6.9, 6.10, 6.11 and 6.12, respectively) numerical results can be obtained by applying the method of moments. In Section 6.13 this method is discussed in some detail.

To have some check on the numerical results one should at least inspect to what degree of accuracy the conservation of energy in the scattered field and/or the field-reciprocity relation pertaining to the scattered field are satisfied. For this purpose we have included a short derivation of the relevant theorems (Sections 6.8 and 6.7, respectively) and their application to plane-wave scattering.

The analysis is carried out in the frequency domain. We restrict our scattering objects to be linear and time invariant. Then, each frequency component of the electromagnetic field scatters independently of any other frequency component that may be present in it. Along similar lines, the scattering of electromagnetic waves in the time domain can be investigated. Much useful material on the application of integral equations to the solution of three-dimensional scattering problems can be found in [6.1]. A discussion of integral-equation methods for transient scattering is found in [6.2].

SI-units are used throughout the chapter.

6.1 The geometry of the configuration

In a medium of infinite extent an object is present whose electromagnetic properties differ from those of its surroundings (Fig.6.1). The object

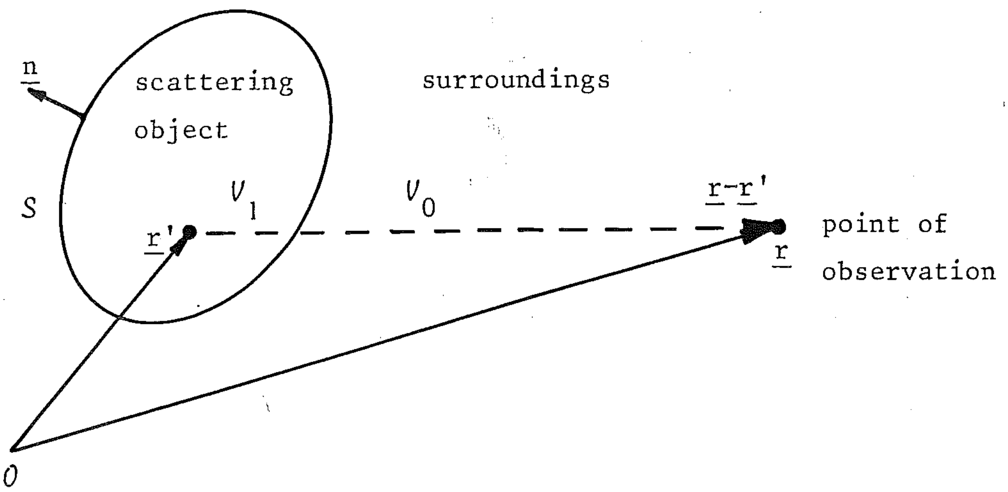

Fig. 6.1. Geometry of the scattering object, with its surroundings and an arbitrary point of observation.

occupies the domain inside a bounded closed surface S. S is assumed to be sufficiently regular, i.e. the unit vector $\underline{n}$ along its normal is a piecewise continuous vector function of position on S. The bounded domain inside S is called V_1, the unbounded domain outside S is called V_0, while we take $\underline{n}$ such that it points away from V_1.

To locate a point in the configuration, we shall in our general considerations employ the orthogonal, Cartesian coordinates x, y, z with respect to a given, orthogonal, Cartesian reference frame. The latter is specified by its origin O and the three mutually perpendicular base vectors of unit length $\underline{i}_x$, $\underline{i}_y$, $\underline{i}_z$. In the given order, the base vectors form a right-handed system. The time coordinate is denoted by t. In specifying the position of an observer we shall often use the position vector

$$\underline{r} = x\underline{i}_x + y\underline{i}_y + z\underline{i}_z. \tag{6.1}$$

Further, the variables of integration over a subdomain of $\mathbb{R}^3$ will often collectively be denoted by

$$\underline{r}' = x'\underline{i}_x + y'\underline{i}_y + z'\underline{i}_z. \tag{6.2}$$

The position of an observer can also be expressed in terms of the distance r from the origin to the point of observation and the unit vector $\underline{\theta}$ pointing from the origin to this point. In terms of $\underline{r}$ we have

$$r = (\underline{r}\cdot\underline{r})^{\frac{1}{2}} = (x^2 + y^2 + z^2)^{\frac{1}{2}}, \tag{6.3}$$

in which $(\ldots)^{\frac{1}{2}} \geq 0$ and

$$\underline{\theta} = \underline{r}/r. \tag{6.4}$$

Obviously, the end point of $\underline{\theta}$ determines the position on the surface Ω of the sphere of unit radius and centre at the origin, i.e.

$$\Omega = \{\underline{r} | \underline{r} \in \mathbb{R}^3, \ \underline{r}\cdot\underline{r} = 1\}. \tag{6.5}$$

The computations related to the integral-equation formulation of scattering problems often require the numerical evaluation of volume integrals over V_1 and/or surface integrals over S. One of the methods to perform the numerical integrations is to subdivide V_1 into a suitable number of tetrahedra and S into the corresponding plane triangles. The maximum diameter of these elements should be so small that with sufficient accuracy V_1 coincides with the union of the solid tetrahedra and S with the union of the plane triangles (Fig. 6.2). As the tetrahedra and the triangles

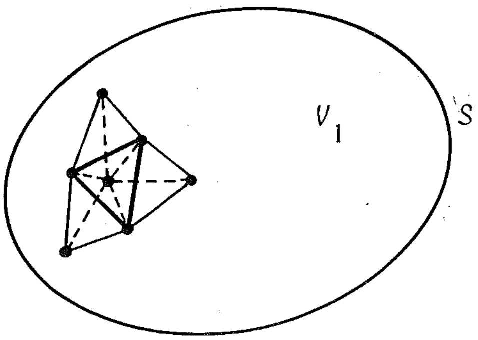

Fig. 6.2. Subdivision of V_1 into tetrahedra and of S into triangles.

need not be mutually identical, it is advantageous to reduce the volume integral over an arbitrary solid tetrahedron and the surface integral over an arbitrary triangle to a suitable standard form.

First, we consider the volume integral

$$I_V = \iiint_{T_V} f(\underline{r})\ dV(\underline{r}), \tag{6.6}$$

where T_V denotes the solid tetrahedron whose corners have the position vectors $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$ and $\underline{r}_4$ respectively and where f denotes some integrable function of position with domain T_V. The desired standard form is arrived at by writing

$$\underline{r} = \lambda(\underline{r}_1-\underline{r}_4) + \mu(\underline{r}_2-\underline{r}_4) + \nu(\underline{r}_3-\underline{r}_4) + \underline{r}_4 \text{ where } 0 < \lambda < 1,$$

$$0 < \mu < 1,\ 0 < \nu < 1 \text{ and } \lambda + \mu + \nu < 1 \text{ for } \underline{r} \in T_V. \tag{6.7}$$

The volume V of T_V can be expressed in terms of the position vectors of its corners as follows:

$$V = (1/6)[\underline{r}_1\cdot(\underline{r}_2\times\underline{r}_3) - \underline{r}_2\cdot(\underline{r}_3\times\underline{r}_4) + \underline{r}_3\cdot(\underline{r}_4\times\underline{r}_1) - \underline{r}_4\cdot(\underline{r}_1\times\underline{r}_2)] \tag{6.8}$$

provided that the corners are numbered as shown in Fig. 6.3 (see Exercise 6.3). With the aid of (6.7) and (6.8) we arrive at (see Exercise 6.4)

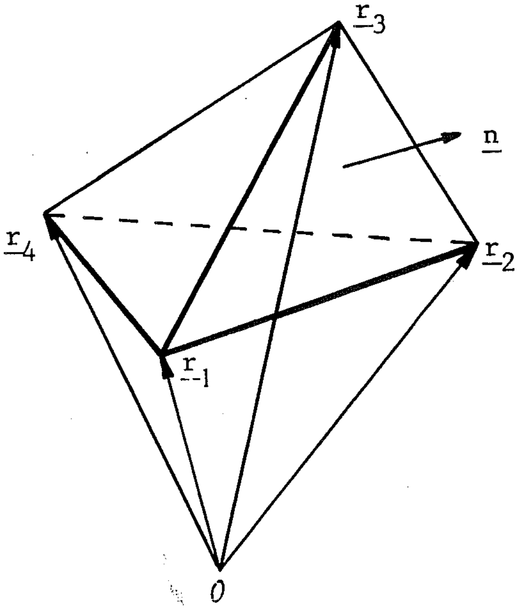

Fig.6.3. Tetrahedron T_V *and numbering of its corners.*

$$\iiint_{T_V} f(\underline{r})\, dV(\underline{r}) = 6V\int_0^1 d\lambda \int_0^{1-\lambda} d\mu$$

$$\int_0^{1-\lambda-\mu} f[\lambda\underline{r}_1+\mu\underline{r}_2+\nu\underline{r}_3 +(1-\lambda-\mu-\nu)\underline{r}_4]\, d\nu. \qquad (6.9)$$

Secondly, we consider the surface integral

$$\underline{I}_A = \iint_{T_A} f(\underline{r})\underline{n}(\underline{r})dA(\underline{r}), \qquad (6.10)$$

where T_A denotes the plane triangle whose corners have the position vectors $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$, respectively, f denotes some integrable function of position with domain T_A and $\underline{n}$ is the unit vector along the normal to T_A as shown in Fig. 6.3. The desired standard form is arrived at by writing

$$\underline{r} = \lambda(\underline{r}_1-\underline{r}_3) + \mu(\underline{r}_2-\underline{r}_3) + \underline{r}_3 \text{ where } 0 < \lambda < 1,\ 0 < \mu < 1 \text{ and}$$

$$\lambda + \mu < 1 \text{ for } \underline{r} \in T_A. \qquad (6.11)$$

The vectorial area $\underline{A}$ of T_A can be expressed in terms of the position vectors of its corners as follows

$$\underline{A} = \tfrac{1}{2}(\underline{r}_1 \times \underline{r}_2 + \underline{r}_2 \times \underline{r}_3 + \underline{r}_3 \times \underline{r}_1) \tag{6.12}$$

(see Exercise 6.5). With the aid of (6.11) and (6.12) we arrive at (see Exercise 6.6)

$$\iint_{T_A} f(\underline{r})\underline{n}(\underline{r})\, dA(\underline{r}) = 2\underline{A}\int_0^1 d\lambda \int_0^{1-\lambda} f[\lambda\underline{r}_1 + \mu\underline{r}_2 + (1-\lambda-\mu)\underline{r}_3]\, d\mu. \tag{6.13}$$

EXERCISES

Exercise 6.1. Let $f = f(\underline{r})$ denote a continuous function of position $\underline{r}$ in $\mathbb{R}^3$, i.e. $f:\mathbb{R}^3 \to \mathbb{R}$ with $\underline{r} \in \mathbb{R}^3$ and let S be the surface $S = \{\underline{r}|f(\underline{r}) = \text{constant}\}$. Assume that f is continuously differentiable with respect to $\underline{r}$ (i.e. with respect to x, y and z). Express the unit vector $\underline{n} = \underline{n}(\underline{r})$, with $\underline{r} \in S$, along the normal to S in terms of the derivatives of f.
Answer: $\underline{n} = \pm\underline{\nabla}f/(\underline{\nabla}f\cdot\underline{\nabla}f)^{\frac{1}{2}}$, where $\underline{\nabla}f = (\partial_x f)\underline{i}_x + (\partial_y f)\underline{i}_y + (\partial_z f)\underline{i}_z$.

Exercise 6.2. Let $\underline{n}$ denote the unit vector along the normal to a certain surface S in $\mathbb{R}^3$ and let $\underline{E}$ denote some (real or complex) vectorial quantity that is defined on S. Determine the expression for (a) the normal part $\underline{E}_{norm}$ of $\underline{E}$ on S, (b) the tangential part $\underline{E}_{tang}$ of $\underline{E}$ on S.
Answer: (a) $\underline{E}_{norm} = (\underline{n}\cdot\underline{E})\underline{n}$; (b) $\underline{E}_{tang} = \underline{E} - \underline{E}_{norm} = \underline{E} - (\underline{n}\cdot\underline{E})\underline{n} = \underline{n}\times(\underline{E}\times\underline{n}) = (\underline{n}\times\underline{E}) \times \underline{n}$.

Exercise 6.3. Let S denote a sufficiently regular, bounded, closed surface in $\mathbb{R}^3$ and let V be the (bounded) domain interior to S. Then, from Gauss' divergence theorem it follows that $\iint_S \underline{n}\cdot\underline{r}\, dA = \iiint_V \underline{\nabla}\cdot\underline{r}\, dV = 3V$, where $\underline{n}$ denotes the unit vector along the normal to S, pointing away from V, and $V = \iiint_V dV$ is the volume of V. Show that application of this result to the tetrahedron shown in Fig. 6.3 leads to Equation (6.8).

Exercise 6.4. Check Equation (6.9) by taking $f(\underline{r}) = 1$ when $\underline{r} \in T_V$ and proving that $\int_0^1 d\lambda \int_0^{1-\lambda} d\mu \int_0^{1-\lambda-\mu} d\nu = 1/6$.

Exercise 6.5. Let T_A be the plane triangle in $\mathbb{R}^3$ used in Equation (6.10). Show that the unit vector $\underline{n}$ along the normal to T_A (see also Fig. 6.3) is given by $\underline{n} = \tfrac{1}{2} A^{-1} [(\underline{r}_2-\underline{r}_1) \times (\underline{r}_3-\underline{r}_1)]$, where A is the area of T_A and derive Equation (6.12) for $\underline{A} = \iint_{T_A} \underline{n}\, dA$.

Exercise 6.6. Check Equation (6.13) by taking $f(\underline{r}) = 1$ when $\underline{r} \in T_A$ and proving that $\int_0^1 d\lambda \int_0^{1-\lambda} d\mu = \frac{1}{2}$.

Exercise 6.7. If $f = f(\underline{r})$ is a linear function of position, the integrand in the right-hand side of Equation (6.9) can be written as $f = \lambda(f_1-f_4) + \mu(f_2-f_4) + \nu(f_3-f_4) + f_4$, where $f_i \overset{\text{def}}{=} f(\underline{r}_i)$ with $i = 1, 2, 3, 4$. Show that in this case we have $\iiint_{T_V} f(\underline{r})\, dV(\underline{r}) = \frac{1}{4}(f_1+f_2+f_3+f_4)\, V$. (This formula is of practical importance to the numerical approximation of the integral in Equation (6.9).)

Exercise 6.8. If $f = f(\underline{r})$ is a linear function of position, the integrand in the right-hand side of Equation (6.13) can be written as $f = \lambda(f_1-f_3) + \mu(f_2-f_3) + f_3$, where $f_i \overset{\text{def}}{=} f(\underline{r}_i)$ with $i = 1, 2, 3$. Show that in this case we have $\iint_{T_A} f(\underline{r})\underline{n}(\underline{r})\, dA(\underline{r}) = (1/3)(f_1+f_2+f_3)\, \underline{A}$. (This formula is of practical importance to the numerical approximation of the integral in Equation (6.13).)

6.2 Description of the electromagnetic field in the configuration

The electromagnetic state in the configuration is characterized by the five vectorial quantities

$\underline{E}$ = electric-field strength,
$\underline{H}$ = magnetic-field strength,
$\underline{J}$ = current density,
$\underline{D}$ = electric-flux density,
$\underline{B}$ = magnetic-flux density

and the scalar quantity

ρ = volume density of charge.

These six quantities will collectively be denoted as the electromagnetic-field quantities. Our analysis will be carried out in the frequency domain and this implies that only a single frequency component of the electromagnetic-field quantities will be considered. Let ω denote the angular frequency of the frequency component under consideration, then we shall denote the complex electromagnetic-field quantities associated with this frequency component by $\{\underline{E}(\underline{r}), \underline{H}(\underline{r}), \underline{J}(\underline{r}), \underline{D}(\underline{r}), \underline{B}(\underline{r}), \rho(\underline{r})\} \exp(-i\omega t)$, where i is the imaginary unit. By superimposing different frequency components we can reconstruct electromagnetic fields that vary in time in a rather arbitrary way (Fourier synthesis). For sinusoidal oscillations with angular frequency ω we simply take the real part of the complex expressions, which results into a time-periodic field with period $T = 2\pi/\omega$.

In a source-free subdomain of $\mathbb{R}^3$ the quantities $\underline{E} = \underline{E}(\underline{r})$, $\underline{H} = \underline{H}(\underline{r})$, $\underline{J} = \underline{J}(\underline{r})$, $\underline{D} = \underline{D}(\underline{r})$, $\underline{B} = \underline{B}(\underline{r})$ and $\rho = \rho(\underline{r})$ satisfy the electromagnetic-field equations in the frequency domain

$$\underline{\nabla} \times \underline{H} = \underline{J} - i\omega\underline{D}, \tag{6.14a}$$

$$\underline{\nabla} \times \underline{E} = i\omega\underline{B}, \tag{6.14b}$$

$$\underline{\nabla}\cdot\underline{D} = \rho, \tag{6.14c}$$

$$\underline{\nabla}\cdot\underline{B} = 0, \tag{6.14d}$$

$$\underline{\nabla}\cdot\underline{J} - i\omega\rho = 0. \tag{6.14e}$$

In analyzing the scattering configuration it is useful to distinguish the field quantities in the scattering object from those in its surroundings. Further, it is useful to consider separately the electromagnetic field that would be present if the scattering object showed no electromagnetic contrast with its surroundings. This field is called the incident field; it is defined in all space. The difference between the field that is actually present in the configuration and the incident field is called the scattered field; this is defined in all space except on $\mathcal{S}$. For the nomenclature regarding the different field constituents we refer to Table 6.1.

Table 6.1. Nomenclature and properties of the different field constituents in the scattering configuration

field quantities	type	defined in	source-free in
$\underline{E}_1, \underline{H}_1, \underline{J}_1, \underline{D}_1, \underline{B}_1, \rho_1$	total field	$\mathcal{V}_1$	$\mathcal{V}_1$
$\underline{E}_0, \underline{H}_0, \underline{J}_0, \underline{D}_0, \underline{B}_0, \rho_0$	total field	$\mathcal{V}_0$	
$\underline{E}^i, \underline{H}^i, \underline{J}^i, \underline{D}^i, \underline{B}^i, \rho^i$	incident field	$\mathbb{R}^3$	$\mathcal{V}_1$
$\underline{E}^s, \underline{H}^s, \underline{J}^s, \underline{D}^s, \underline{B}^s, \rho^s$	scattered field	$\mathbb{R}^3\backslash\mathcal{S}$	$\mathcal{V}_0$

$\mathcal{V}_1$ = scattering object, $\mathcal{V}_0$ = surroundings; total field = incident field + scattered field.

Due to the electromagnetic contrast between the scattering object and its surroundings the electromagnetic properties of the configuration in general change abruptly when crossing S. Hence, the electromagnetic-field equations have to be supplemented by boundary conditions on S. As far as the boundary conditions are concerned we distinguish between three types of scattering objects: (a) electromagnetically penetrable objects, (b) electrically impenetrable (i.e. perfectly conducting) objects, (c) perfectly conducting objects of vanishing thickness (screens). The relevant boundary conditions and some of their corollaries are summarized below.

(a) Electromagnetically penetrable object

The scattering object is called electromagnetically penetrable if the electromagnetic field can penetrate into the object. Then the tangential parts of the electric-field strength and the magnetic-field strength are continuous across S, i.e.

$$\underline{n} \times \underline{E}_0 = \underline{n} \times \underline{E}_1 \text{ and } \underline{n} \times \underline{H}_0 = \underline{n} \times \underline{H}_1 \text{ when } \underline{r} \in S. \tag{6.15}$$

In this case we assume that no perfectly conducting material is present, which implies that neither surface current nor surface charge is present on S.

(b) Electrically impenetrable (i.e. perfectly conducting) object

The scattering object is called electrically impenetrable or perfectly conducting if the tangential part of the electric-field strength vanishes upon approaching S through V_0, i.e.

$$\underline{n} \times \underline{E}_0 = \underline{0} \text{ when } \underline{r} \in S. \tag{6.16}$$

In this case, surface current and surface charge are, in general, present on S. They follow from

$$\underline{n} \times \underline{H}_0 = \underline{J}_S \text{ when } \underline{r} \in S \tag{6.17}$$

and

$$\underline{n}\cdot\underline{D}_0 = \rho_S \text{ when } \underline{r} \in S, \tag{6.18}$$

where

$\underline{J}_S$ = surface-current density,
ρ_S = surface-charge density.

Note, that $\underline{n}\cdot\underline{J}_S = 0$.

(c) Perfectly conducting object of vanishing thickness (screen)

If the scattering object is of vanishing thickness it is called a screen. A screen is a two-sided surface. Its two faces are called S^- and S^+, respectively, its boundary curve will be called C (Fig.6.4). If, further, the screen is perfectly conducting the tangential part of the electric-field

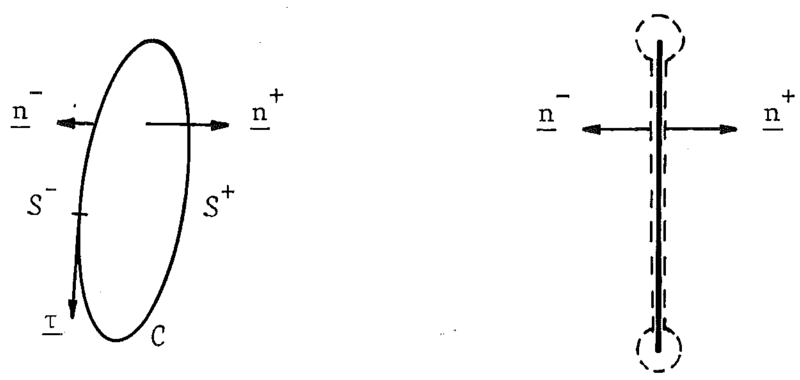

Fig.6.4. Scattering object of vanishing thickness with its two faces S^- and S^+ and its boundary curve C.

strength vanishes on both faces of the screen, i.e.

$$\underline{n}^- \times \underline{E}_0 = \underline{0} \text{ when } \underline{r} \in S^- \text{ and}$$

$$\underline{n}^+ \times \underline{E}_0 = \underline{0} \text{ when } \underline{r} \in S^+. \tag{6.19}$$

On both faces, surface current as well as surface charge is present according to

$$\underline{n}^{-} \times \underline{H}_0 = \underline{J}_S^{-} \text{ when } \underline{r} \in S^{-}, \tag{6.20a}$$

$$\underline{n}^{+} \times \underline{H}_0 = \underline{J}_S^{+} \text{ when } \underline{r} \in S^{+}, \tag{6.20b}$$

and

$$\underline{n}^{-} \cdot \underline{D}_0 = \rho_S^{-} \text{ when } \underline{r} \in S^{-}, \tag{6.21a}$$

$$\underline{n}^{+} \cdot \underline{D}_0 = \rho_S^{+} \text{ when } \underline{r} \in S^{+}. \tag{6.21b}$$

On C no charge can accumulate, since that would lead to a too singular behaviour of the electromagnetic field in the neighbourhood of C (a charged line has no finite energy). Let $\underline{\tau}$ denote the unit vector along the tangent to C (Fig.6.4), then this condition can be expressed in terms of the surface current density as

$$(\underline{n}^{-} \times \underline{\tau}) \cdot \underline{J}_S^{-} \to 0 \quad \text{as } \underline{r} \to C, \tag{6.22a}$$

$$(\underline{n}^{+} \times \underline{\tau}) \cdot \underline{J}_S^{+} \to 0 \quad \text{as } \underline{r} \to C. \tag{6.22b}$$

Equation (6.22) is a weak form of the so-called edge condition. A more detailed analysis leads to a specification of the local behaviour of the electromagnetic-field components in the neighbourhood of the edge of a perfectly conducting screen. For details, we refer to [6.3].

The next point to be discussed is the specification of the electromagnetic properties of the media that are present in the configuration. In order that the frequency-domain analysis can be carried out for each frequency separately, it is necessary that the media are linear and time-invariant. Further, the medium surrounding the scattering object will be taken as simple as possible, which means locally reacting, isotropic, homogeneous and lossless. Then, its electromagnetic behaviour is characterized by constitutive relations of the type

$$\underline{J}_0 = \underline{0},\ \underline{D}_0 = \varepsilon_0 \underline{E}_0,\ \underline{B}_0 = \mu_0 \underline{H}_0$$
$$\text{when } \underline{r} \in \{\text{source-free subdomain of } V_0\}, \tag{6.23}$$

in which the permittivity ε_0 and the permeability μ_0 have real, constant values that are also assumed to be positive. On account of the first equation of (6.23) we then also have

$$\rho_0 = 0 \text{ when } \underline{r} \in \{\text{source-free subdomain of } V_0\}. \tag{6.24}$$

In particular, (6.23) and (6.24) hold for the scattered field and as the latter has no sources in V_0 we have

$$\underline{J}^s = \underline{0},\ \underline{D}^s = \varepsilon_0\underline{E}^s,\ \underline{B}^s = \mu_0\underline{H}^s,\ \rho^s = 0 \quad \text{when } \underline{r} \in V_0. \tag{6.25}$$

As to the electromagnetic properties of the scattering object we will additionally only assume that it is locally reacting and leave open the possibility of its being anisotropic, inhomogeneous and lossy. The constitutive relations are then of the type

$$\begin{bmatrix} \underline{J}_1 \\ \underline{D}_1 \\ \underline{B}_1 \end{bmatrix} = \begin{bmatrix} \text{consti-} \\ \text{tutive} \\ \text{matrix} \end{bmatrix} \begin{bmatrix} \underline{E}_1 \\ \underline{H}_1 \end{bmatrix} \quad \text{when } \underline{r} \in V_1, \tag{6.26}$$

where the constitutive matrix is a 3 × 2 matrix of the constitutive tensors of rank two that describe the electromagnetic behaviour of the material. On several occasions it is advantageous to use the <u>electromagnetic contrast</u> that the scattering object shows with respect to its surroundings. This contrast is characterized by the contrast current density $\underline{J}_1$ (note that $\underline{J}_0 = \underline{0}$), the electric contrast polarization

$$\underline{P}_1 \stackrel{\text{def}}{=\!=} \underline{D}_1 - \varepsilon_0\underline{E}_1 \quad \text{when } \underline{r} \in V_1 \tag{6.27}$$

and the contrast magnetization

$$\underline{M}_1 \stackrel{\text{def}}{=\!=} \underline{B}_1/\mu_0 - \underline{H}_1 \text{ when } \underline{r} \in V_1. \tag{6.28}$$

Finally, it is observed that our formulation of the problem includes the case where any piece of linear, time-invariant, locally reacting matter is present in empty space. In this case the values of ε_0 and μ_0 reduce to the values of the permittivity and the permeability in vacuo, respectively.

Any sinusoidally in time varying vectorial quantity (such as the electric-field strength and the magnetic-field strength) has a certain state of polarization. The latter can be either elliptical, circular or linear and can easily be inferred from the complex representation of the relevant field quantity. The criteria pertaining to the electric-field strength are listed in Table 6.2.

Table 6.2. State of polarization of a sinusoidally in time varying vectorial field quantity. (The criteria shown are those for the electric-field strength.)

state of polarization	criterion
elliptic	in general
circular	$\underline{E}\cdot\underline{E} = 0$
linear	$\underline{E} \times \underline{E}^{*} = \underline{0}$

The sinusoidally in time varying electric-field strength is given by $\mathrm{Re}[\underline{E}(\underline{r})\exp(-i\omega t)]$; * denotes complex conjugate.

EXERCISES

Exercise 6.9. Show that $\underline{\nabla}\cdot(\underline{J}-i\omega\underline{D}) = 0$ and explain that this result is compatible with Eq.(6.14a).

Exercise 6.10. Explain that Eq.(6.14d) is compatible with Eq.(6.14b).

Exercise 6.11. Derive from Eqs.(6.14a) and (6.14b) the relation $\underline{\nabla}\cdot(\underline{E}\times\underline{H}^{*}) = -[\underline{E}\cdot\underline{J}^{*} + i\omega\underline{E}\cdot\underline{D}^{*} - i\omega\underline{H}^{*}\cdot\underline{B}]$ when $\underline{r} \in$ {source-free subdomain of $\mathbb{R}^3$}. (This relation is known as the electromagnetic complex power balance.)

Exercise 6.12. Let $\underline{S} = \mathrm{Re}[\underline{E}(\underline{r})\exp(-i\omega t)] \times \mathrm{Re}[\underline{H}(\underline{r})\exp(-i\omega t)]$ be the power flow density (Poynting vector) of a sinusoidally in time varying electromagnetic field. Show that the time average over a single period $T = 2\pi/\omega$ of $\underline{S}$ is given by $\langle\underline{S}\rangle_T = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^{*})$.

Exercise 6.13. Use Eq.(6.15) to show that $\underline{n}\cdot(\underline{E}\times\underline{H}^*)$ is continuous across the boundary of an electromagnetically penetrable scattering object. (Hint: observe that $\underline{n}\cdot(\underline{E}\times\underline{H}^*) = \underline{H}^*\cdot(\underline{n}\times\underline{E}) = \underline{E}\cdot(\underline{H}^*\times\underline{n})$.)

Exercise 6.14. Use Eq.(6.16) to show that $\underline{n}\cdot(\underline{E}\times\underline{H}^*)$ vanishes on the boundary of an electrically impenetrable (i.e. perfectly conducting) scattering object.

Exercise 6.15. Determine the constitutive matrix (cf.Eq.(6.26)) of the medium present in the scattering object when this is isotropic with scalar conductivity $\sigma_1 = \sigma_1(\underline{r})$, scalar permittivity $\varepsilon_1 = \varepsilon_1(\underline{r})$ and scalar permeability $\mu_1 = \mu_1(\underline{r})$.

Answer:

$$\begin{bmatrix} \underline{J}_1 \\ \underline{D}_1 \\ \underline{B}_1 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 \\ \varepsilon_1 & 0 \\ 0 & \mu_1 \end{bmatrix} \begin{bmatrix} \underline{E}_1 \\ \underline{H}_1 \end{bmatrix} .$$

Exercise 6.16. Show that if the medium of Exercise 6.15 is dissipative, we have $\text{Re}(\sigma_1) > 0$, $\text{Im}(\varepsilon_1) > 0$ and $\text{Im}(\mu_1) > 0$ at any $\underline{r} \in V_1$. (Hint: use the results of Exercises 6.11 and 6.12 and observe that $\langle\underline{\nabla}\cdot\underline{S}\rangle_T < 0$ for a dissipative medium.)

Exercise 6.17. Show that if the medium of Exercise 6.15 is lossless, we have $\text{Re}(\sigma_1) = 0$, $\text{Im}(\varepsilon_1) = 0$ and $\text{Im}(\mu_1) = 0$ at any $\underline{r} \in V_1$. (Hint: use the results of Exercises 6.11 and 6.12 and observe that $\langle\underline{\nabla}\cdot\underline{S}\rangle_T = 0$ for a lossless medium.)

Exercise 6.18. Determine the state of polarization of the sinusoidally in time varying electric-field strength $\text{Re}[\underline{E}(\underline{r})\exp(-i\omega t)]$ when $\underline{E} = \underline{e}\exp(i\psi)$, where $\underline{e}$ and ψ are real.

Answer: linear.

6.3 The plane wave as incident field

In many applications the incident field can, at least in the neighbourhood of the scattering object, be approximated by a plane wave. Therefore, the case of an incident plane wave is often considered in the theoretical analysis of scattering problems. For generality as well as in view of some applications in the theory of diffraction radiation from a charged particle [6.4], we shall include non-uniform plane waves in the discussion. Let $\underline{\alpha}$ denote the complex unit vector in the "direction of propagation" of the wave, then its electric- and its magnetic-field strengths are written as

$$\{\underline{E}^i, \underline{H}^i\} = \{\underline{e}^i_{\underline{\alpha}}, \underline{h}^i_{\underline{\alpha}}\} \exp(ik_0\underline{\alpha}\cdot\underline{r}) \quad \text{when } \underline{r} \in \mathbb{R}^3, \tag{6.29}$$

where $\underline{e}^i_{\underline{\alpha}}$ and $\underline{h}^i_{\underline{\alpha}}$ are constant complex vectors,

$$k_0 = \omega(\varepsilon_0\mu_0)^{\frac{1}{2}} \tag{6.30}$$

and

$$\underline{\alpha}\cdot\underline{\alpha} = 1. \tag{6.31}$$

Substitution of (6.29) in (6.14), together with (6.23) and (6.24), leads to the following relations between $\underline{\alpha}$, $\underline{e}^i_{\underline{\alpha}}$ and $\underline{h}^i_{\underline{\alpha}}$:

$$Z_0\underline{\alpha} \times \underline{h}^i_{\underline{\alpha}} = -\underline{e}^i_{\underline{\alpha}}, \tag{6.32a}$$

$$Y_0\underline{\alpha} \times \underline{e}^i_{\underline{\alpha}} = \underline{h}^i_{\underline{\alpha}}, \tag{6.32b}$$

$$\underline{\alpha}\cdot\underline{e}^i_{\underline{\alpha}} = 0, \tag{6.32c}$$

$$\underline{\alpha}\cdot\underline{h}^i_{\underline{\alpha}} = 0, \tag{6.32d}$$

where

$$Z_0 = (\mu_0/\varepsilon_0)^{\frac{1}{2}} \tag{6.33}$$

and

$$Y_0 = (\varepsilon_0/\mu_0)^{\frac{1}{2}}. \tag{6.34}$$

The quantity $Z_0\underline{\alpha}$ can be regarded as the vectorial wave impedance of the plane wave and the quantity $Y_0\underline{\alpha}$ as its vectorial wave admittance.

Next, we investigate the time-averaged power flow density $<\underline{S}^i_{\underline{\alpha}}>_T$ in the sinusoidally in time varying plane wave $\mathrm{Re}[\{\underline{E}^i,\underline{H}^i\}\exp(-i\omega t)]$. We obtain (cf.Exercise 6.12)

$$<\underline{S}^i_{\underline{\alpha}}>_T = \tfrac{1}{2}\,\mathrm{Re}(\underline{e}^i_{\underline{\alpha}} \times \underline{h}^{i\,*}_{\underline{\alpha}})\,\exp[-2\,\mathrm{Im}(\underline{\alpha})\cdot\underline{r}]. \qquad (6.35)$$

For a uniform plane wave, $\underline{\alpha}$ is a real unit vector. Then, the plane wave is transverse with respect to its (real) direction of propagation, while its time-averaged power flow density is a constant vector given by

$$<\underline{S}^i_{\underline{\alpha}}>_T = \tfrac{1}{2}\,\mathrm{Re}(\underline{e}^i_{\underline{\alpha}} \times \underline{h}^{i\,*}_{\underline{\alpha}}) \quad \text{(uniform plane wave).} \qquad (6.36)$$

On account of (6.32) alternative expressions for $<\underline{S}^i_{\underline{\alpha}}>_T$ are in this case given by

$$<\underline{S}^i_{\underline{\alpha}}>_T = \tfrac{1}{2}\,Y_0(\underline{e}^i_{\underline{\alpha}}\cdot\underline{e}^{i\,*}_{\underline{\alpha}})\,\underline{\alpha} \quad \text{(uniform plane wave),} \qquad (6.37)$$

or

$$<\underline{S}^i_{\underline{\alpha}}>_T = \tfrac{1}{2}\,Z_0(\underline{h}^i_{\underline{\alpha}}\cdot\underline{h}^{i\,*}_{\underline{\alpha}})\,\underline{\alpha} \quad \text{(uniform plane wave).} \qquad (6.38)$$

EXERCISES

Exercise 6.19. Derive the relations of Eq.(6.32).

Exercise 6.20. Prove that Eqs.(6.36), (6.37) and (6.38) hold for a uniform plane wave.

Exercise 6.21. Substitute $\underline{\alpha} = \mathrm{Re}(\underline{\alpha}) + i\,\mathrm{Im}(\underline{\alpha})$ in Eq.(6.31) and show that from the resulting equation it follows that $\mathrm{Re}(\underline{\alpha})\cdot\mathrm{Re}(\underline{\alpha}) - \mathrm{Im}(\underline{\alpha})\cdot\mathrm{Im}(\underline{\alpha}) = 1$ and $\mathrm{Re}(\underline{\alpha})\cdot\mathrm{Im}(\underline{\alpha}) = 0$.

6.4 Properties of the scattered field in the far-field region

The unbounded subdomain of V_0 where all points are at a large distance from the boundary surface S of the scattering object is called the far-field region of the scattering configuration. The distance from a point of observation to a point of the scattering object is called large if it is large compared with (a) the maximum diameter of the scattering object, (b) the wavelength of electromagnetic radiation in the medium surrounding the scattering object. (This wavelength is given by $\lambda_0 = 2\pi c_0/\omega$, with $c_0 = (\varepsilon_0\mu_0)^{-\frac{1}{2}}$.) It can be shown (for a proof, see Section 6.6) that in the far-field region the main contribution to the scattered field consists of a radially propagating, expanding, spherical wave with an angularly dependent amplitude. This spherical wave is called the far-field approximation of the scattered field. The fact that the spherical wave is an expanding one (and not a contracting one) is in accordance with the principle of causality. The electric- and the magnetic-field strengths of the scattered field in the far-field region are written as

$$\{\underline{E}^s(\underline{r}),\ \underline{H}^s(\underline{r})\} \sim \{\underline{e}^s(\underline{\theta}),\ \underline{h}^s(\underline{\theta})\}\exp(ik_0r)/4\pi r \quad \text{as } r \to \infty, \tag{6.39}$$

in which r is given by (6.3), $\underline{\theta}$ by (6.4) and k_0 by (6.30). The angularly dependent spherical-wave amplitudes $\underline{e}^s$ and $\underline{h}^s$ satisfy the following relations

$$Z_0\underline{\theta} \times \underline{h}^s = -\underline{e}^s, \tag{6.40a}$$

$$Y_0\underline{\theta} \times \underline{e}^s = \underline{h}^s, \tag{6.40b}$$

$$\underline{\theta}\cdot\underline{e}^s = 0, \tag{6.40c}$$

$$\underline{\theta}\cdot\underline{h}^s = 0, \tag{6.40d}$$

where Z_0 is given by (6.33) and Y_0 by (6.34). Equation (6.40) can be inferred from (6.14), together with (6.25), if (6.39) is substituted in these equations and only terms of order $O(r^{-1})$ are retained. Comparison of (6.40) with (6.32) shows that the properties of $\underline{e}^s$ and $\underline{h}^s$ are the same as those pertaining to a uniform plane wave propagating along $\underline{\theta}$.

Next, we investigate the time-averaged power flow density $\langle\underline{S}^s\rangle_T$ in the sinusoidally in time varying scattered field $\mathrm{Re}[\{\underline{E}^s, \underline{H}^s\}\exp(-i\omega t)]$. We obtain (cf. Exercise 6.12)

$$<\underline{S}^s>_T = \tfrac{1}{2}\,\mathrm{Re}(\underline{E}^s \times \underline{H}^{s*}). \qquad (6.41)$$

Substitution of (6.39) in the right-hand side of (6.41) leads to the expression for $<\underline{S}^s>_T$ in the far-field region

$$<\underline{S}^s>_T \sim \tfrac{1}{2}\,\mathrm{Re}(\underline{e}^s \times \underline{h}^{s*})/(4\pi r)^2 \quad \text{as } r \to \infty. \qquad (6.42)$$

On account of (6.40), alternative expressions for $<\underline{S}^s>_T$ in the far-field region are

$$<\underline{S}^s>_T \sim \tfrac{1}{2}\,Y_0(\underline{e}^s\cdot\underline{e}^{s*})\underline{\theta}/(4\pi r)^2 \quad \text{as } r \to \infty \qquad (6.43)$$

and

$$<\underline{S}^s>_T \sim \tfrac{1}{2}\,Z_0(\underline{h}^s\cdot\underline{h}^{s*})\underline{\theta}/(4\pi r)^2 \quad \text{as } r \to \infty. \qquad (6.44)$$

Equations (6.43) and (6.44) show that in the far-field region the time-averaged power flow density of the scattered wave is an outward radial vector, i.e. $\underline{\theta}\cdot<\underline{S}^s>_T$ is a non-negative quantity.

We now introduce the radiation intensity I^s of the scattered wave through the relation

$$<\underline{S}^s>_T \sim I^s(\underline{\theta})\underline{\theta}/r^2 \quad \text{as } r \to \infty, \text{ for any } \underline{\theta} \in \Omega. \qquad (6.45)$$

The SI-unit of I^s is watt/steradian (W/sr). On account of (6.42), (6.43) and (6.44), I^s is given by either of the expressions

$$I^s = (1/32\pi^2)\,\mathrm{Re}[(\underline{e}^s \times \underline{h}^{s*})\cdot\underline{\theta}]$$

$$= (1/32\pi^2)\,Y_0(\underline{e}^s\cdot\underline{e}^{s*})$$

$$= (1/32\pi^2)\,Z_0(\underline{h}^s\cdot\underline{h}^{s*}). \qquad (6.46)$$

The time-averaged total power $<P^s>_T$ carried by the scattered field across a closed surface completely surrounding the scattering object is given by

$$\langle P^s\rangle_T = \iint_{\substack{\text{closed surface}\\ \text{completely surround-}\\ \text{ing the obstacle}}} \underline{n}\cdot\langle \underline{S}^s\rangle_T \, dA, \tag{6.47}$$

where $\underline{n}$ denotes the unit vector along the outward normal to the surface of integration. Since the medium in V_0 is assumed to be lossless, the result obtained from (6.47) is independent of the particular surface that is selected to carry out the integration. Two choices are of special interest. In the first, we take the boundary surface of the scattering object as surface of integration and obtain

$$\langle P^s\rangle_T = \iint_S \underline{n}\cdot\langle \underline{S}^s\rangle_T \, dA. \tag{6.48}$$

In the second, we take as surface of integration a sphere with centre at the origin and located in the far-field region. On account of (6.45) we can write the result as

$$\langle P^s\rangle_T = \iint_\Omega I^s \, d\Omega, \tag{6.49}$$

in which one of the expressions (6.46) for I^s can be substituted.

In case the incident field is a uniform plane wave propagating in the direction of the unit vector $\underline{\alpha}$ (cf.Section 6.3), it is customary to introduce the (plane-wave) scattering cross-section $\sigma^s(\underline{\theta},\underline{\alpha})$, given by

$$\sigma^s(\underline{\theta},\underline{\alpha}) \stackrel{\text{def}}{=} 4\pi \; I^s(\underline{\theta})/\underline{\alpha}\cdot\langle \underline{S}^i_{\underline{\alpha}}\rangle_T \qquad \text{with } \underline{\theta}\in\Omega \text{ and } \underline{\alpha}\in\Omega, \tag{6.50}$$

as the quantity that typically compares the power flow in the scattered field in the different directions with the power flow in the incident wave. In (6.50), I^s is given by (6.46), while $\underline{\alpha}\cdot\langle \underline{S}^i_{\underline{\alpha}}\rangle_T$ directly follows from (6.36), (6.37) or (6.38). The SI-unit of σ^s is square metre (m^2).

In several relations, the average of $\sigma^s(\underline{\theta},\underline{\alpha})$ taken over all directions of observation occurs. Denoting this quantity by $\sigma^s_{\underline{\alpha}}$, we have

$$\sigma^s_{\underline{\alpha}} \stackrel{\text{def}}{=} (4\pi)^{-1}\iint_\Omega \sigma^s(\underline{\theta},\underline{\alpha})\, d\Omega(\underline{\theta}). \tag{6.51}$$

On account of (6.49) and (6.50), $\sigma^s_{\underline{\alpha}}$ is also given by

$$\sigma^s_{\underline{\alpha}} = \langle P^s \rangle_T / \underline{\alpha} \cdot \langle \underline{S}^i_{\underline{\alpha}} \rangle_T . \qquad (6.52)$$

EXERCISES

Exercise 6.22. Derive Eq.(6.40) along the lines indicated in the text.

Exercise 6.23. Prove from Eqs. (6.37), (6.38), (6.46) and (6.50) that
$\sigma^s(\underline{\theta},\underline{\alpha}) = (4\pi)^{-1}[\underline{e}^s(\underline{\theta})\cdot\underline{e}^{s*}(\underline{\theta})]/(\underline{e}^i_{\underline{\alpha}}\cdot\underline{e}^{i*}_{\underline{\alpha}}) = (4\pi)^{-1}[\underline{h}^s(\underline{\theta})\cdot\underline{h}^{s*}(\underline{\theta})]/(\underline{h}^i_{\underline{\alpha}}\cdot\underline{h}^{i*}_{\underline{\alpha}})$
and verify that the right-hand sides indeed are expressed in metre^2.

6.5 Source representations for the electromagnetic-field quantities

The basic tool in the integral-equation formulation of scattering problems is a certain integral relation that, for points of observation located in a certain domain in space, leads to a source-type of integral representation for the electromagnetic-field quantities. The present section is devoted to a discussion of integral relations of this kind.

We start with the situation where the electromagnetic-field quantities are defined in a bounded subdomain V_1 of $\mathbb{R}^3$. The boundary of V_1 is the closed surface S (Fig.6.5). S is assumed to be sufficiently regular, i.e. the unit vector $\underline{n}$ along its outward normal is a piecewise continuous vector function of position. The unbounded domain exterior to S is called V_0. In

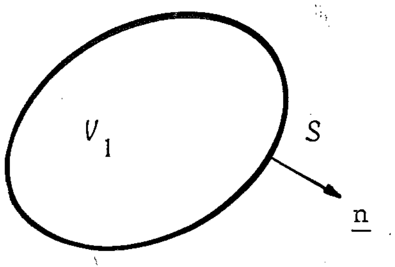

Fig.6.5. The bounded domain V_1 interior to the closed surface S, for which source-type of integral representations of the electromagnetic-field quantities are derived.

view of later applications we assume that sources are present in V_1 whose action can be accounted for by volume-source distributions. The volume densities of the "source currents" appear as source terms in the electromagnetic-field equations, which are written as

$$\underline{\nabla} \times \underline{H} - \underline{J} + i\omega\underline{D} = \underline{J}^e{}_V \quad \text{when } \underline{r} \in V_1, \tag{6.53a}$$

$$\underline{\nabla} \times \underline{E} - i\omega\underline{B} = -\underline{J}^m{}_V \quad \text{when } \underline{r} \in V_1, \tag{6.53b}$$

where

$\underline{J}^e{}_V$ = volume-source density of electric current,

$\underline{J}^m{}_V$ = volume-source density of magnetic current.

Taking the spatial Fourier transform (6.A1) of (6.53a) and (6.53b) and applying (6.A5) we obtain the Fourier transformed electromagnetic-field equations pertaining to the field defined in V_1

$$i\underline{k} \times \underline{\tilde{H}} - \underline{\tilde{J}} + i\omega\underline{\tilde{D}} = \underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S, \tag{6.54a}$$

$$i\underline{k} \times \underline{\tilde{E}} - i\omega\underline{\tilde{B}} = -\underline{\tilde{J}}^m{}_V - \underline{\tilde{J}}^m{}_S, \tag{6.54b}$$

in which $\underline{\tilde{J}}^e{}_S$ and $\underline{\tilde{J}}^m{}_S$ are the Fourier transforms over S of the quantities

$$\underline{J}^e{}_S = -\,\underline{n} \times \underline{H} \quad \text{when } \underline{r} \in S \tag{6.55}$$

and

$$\underline{J}^m{}_S = \underline{n} \times \underline{E} \quad \text{when } \underline{r} \in S, \tag{6.56}$$

respectively. Note that in the right-hand sides of (6.55) and (6.56) the limiting values of the quantities upon approaching S via V_1 are to be taken. The structure of (6.54a) and (6.54b) leads to the interpretation

$\underline{J}^e{}_S$ = surface-source density of electric current,

$\underline{J}^m{}_S$ = surface-source density of magnetic current.

In order to solve $\underline{\tilde{E}}$ and $\underline{\tilde{H}}$ from (6.54a) and (6.54b), assuming that the right-hand sides of these equations are known, we must have the constitutive relations at our disposal. To illustrate the further method, we shall proceed with the simple case where

$$\underline{J} = \sigma \underline{E}, \tag{6.57a}$$

$$\underline{D} = \varepsilon \underline{E}, \tag{6.57b}$$

$$\underline{B} = \mu \underline{H}, \tag{6.57c}$$

in which the conductivity σ, the permittivity ε and the permeability μ are constants, but may have complex values. Taking the spatial Fourier transform of (6.57a)-(6.57c) we obviously arrive at

$$\underline{\tilde{J}} = \sigma \underline{\tilde{E}}, \tag{6.58a}$$

$$\underline{\tilde{D}} = \varepsilon \underline{\tilde{E}}, \tag{6.58b}$$

$$\underline{\tilde{B}} = \mu \underline{\tilde{H}}. \tag{6.58c}$$

Substitution of (6.58a)-(6.58c) in (6.54a) and (6.54b) leads to

$$i\underline{k} \times \underline{\tilde{H}} - (\sigma - i\omega\varepsilon)\underline{\tilde{E}} = \underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S, \tag{6.59a}$$

$$i\underline{k} \times \underline{\tilde{E}} - i\omega\mu\underline{\tilde{H}} = -(\underline{\tilde{J}}^m{}_V + \underline{\tilde{J}}^m{}_S). \tag{6.59b}$$

From these equations we want to solve $\underline{\tilde{E}}$ and $\underline{\tilde{H}}$. To determine $\underline{\tilde{E}}$ we eliminate $\underline{\tilde{H}}$ from (6.59a) and (6.59b), use the vectorial identity

$$i\underline{k} \times (i\underline{k} \times \underline{\tilde{E}}) = i\underline{k}(i\underline{k}\cdot\underline{\tilde{E}}) + (\underline{k}\cdot\underline{k})\,\underline{\tilde{E}} \tag{6.60}$$

and observe that from (6.59a) it follows that

$$-(\sigma - i\omega\varepsilon)\, i\underline{k}\cdot\underline{\tilde{E}} = i\underline{k}\cdot(\underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S). \tag{6.61}$$

We then arrive at

$$[\underline{k}\cdot\underline{k} - (\sigma - i\omega\varepsilon)i\omega\mu]\underline{\tilde{E}} = i\omega\mu(\underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S)$$

$$+ (\sigma - i\omega\varepsilon)^{-1}\, i\underline{k}[i\underline{k}\cdot(\underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S)] - i\underline{k} \times (\underline{\tilde{J}}^m{}_V + \underline{\tilde{J}}^m{}_S). \tag{6.62}$$

A similar procedure leads to

$$[\underline{k}\cdot\underline{k} - (\sigma - i\omega\varepsilon)i\omega\mu]\underline{\tilde{H}} = -(\sigma - i\omega\varepsilon)(\underline{\tilde{J}}^m{}_V + \underline{\tilde{J}}^m{}_S)$$

$$- (i\omega\mu)^{-1}\ i\underline{k}[i\underline{k}\cdot(\underline{\tilde{J}}^m{}_V + \underline{\tilde{J}}^m{}_S)] + i\underline{k} \times (\underline{\tilde{J}}^e{}_V + \underline{\tilde{J}}^e{}_S). \qquad (6.63)$$

The expressions that result when Fourier inversion is applied to the right--hand sides of (6.62) and (6.63) are fairly complicated. Their structure is made somewhat more transparant if judiciously chosen auxiliary quantities are introduced. In this respect we consider the functions

$$\tilde{G} \stackrel{\text{def}}{=} [\underline{k}\cdot\underline{k} - (\sigma - i\omega\varepsilon)i\omega\mu]^{-1} \qquad (6.64)$$

and

$$\underline{\tilde{\Pi}}^{e,m}_{V,S} \stackrel{\text{def}}{=} \tilde{G}\ \underline{\tilde{J}}^{e,m}_{V,S} \qquad (6.65)$$

that occur in the expressions for $\underline{\tilde{E}}$ and $\underline{\tilde{H}}$. With the aid of (6.64) and (6.65) we can write

$$\underline{\tilde{E}} = i\omega\mu(\underline{\tilde{\Pi}}^e{}_V + \underline{\tilde{\Pi}}^e{}_S) + (\sigma - i\omega\varepsilon)^{-1}\ i\underline{k}[i\underline{k}\cdot(\underline{\tilde{\Pi}}^e{}_V + \underline{\tilde{\Pi}}^e{}_S)]$$

$$- i\underline{k} \times (\underline{\tilde{\Pi}}^m{}_V + \underline{\tilde{\Pi}}^m{}_S) \qquad (6.66)$$

and

$$\underline{\tilde{H}} = -(\sigma - i\omega\varepsilon)(\underline{\tilde{\Pi}}^m{}_V + \underline{\tilde{\Pi}}^m{}_S) - (i\omega\mu)^{-1}\ i\underline{k}[i\underline{k}\cdot(\underline{\tilde{\Pi}}^m{}_V + \underline{\tilde{\Pi}}^m{}_S)]$$

$$+ i\underline{k} \times (\underline{\tilde{\Pi}}^e{}_V + \underline{\tilde{\Pi}}^e{}_S). \qquad (6.67)$$

The expression for $\tilde{G}$ arises from the application of a spatial Fourier transform to the inhomogeneous Helmholtz equation

$$[\underline{\nabla}\cdot\underline{\nabla} + (\sigma - i\omega\varepsilon)i\omega\mu]G = -\delta(\underline{r}), \qquad (6.68)$$

where $\delta(\underline{r})$ denotes the three-dimensional unit pulse operating at $\underline{r} = \underline{0}$, provided that the Fourier transform (6.A1) is extended over all space and the contribution from the "sphere at infinity" vanishes in (6.A5). Either from standard theory pertaining to (6.68) or by evaluating the Fourier inversion integral with the right-hand side of (6.64) as spatial Fourier transform,we obtain

$$G = (4\pi|\underline{r}|)^{-1} \exp[i\{(\sigma - i\omega\varepsilon)i\omega\mu\}^{\frac{1}{2}}|\underline{r}|] \text{ for all } \underline{r} \neq \underline{0}, \tag{6.69}$$

where $\mathrm{Re}\{\ldots\}^{\frac{1}{2}} > 0$ and $\mathrm{Im}\{\ldots\}^{\frac{1}{2}} \geq 0$ for the medium under consideration, and $\omega > 0$. The right-hand side of (6.69) is a scalar spherical wave expanding from the point source; it is also called the free-space Green's function of the three-dimensional scalar Helmholtz equation. Note that $G(\underline{r})$ shows an exponential decay as $|\underline{r}| \to \infty$ as long as $\sigma > 0$, and hence the procedure of applying the spatial Fourier transform to (6.68) is justified.

Next we investigate the expressions for the vector potentials that have been defined in (6.65). Since they are the products of two spatial Fourier transforms, they correspond with a convolution in $\underline{r}$-space. We observe that the factors $\tilde{\underline{J}}_V^{e,m}$ arise from a Fourier transform extended over V_1, the factors $\tilde{\underline{J}}_S^{e,m}$ from a Fourier transform extended over S, while $\tilde{G}$ is the Fourier transform of G extended over all space. Consequently, we should identify f in (6.A8) with $\underline{J}$ and g with G. This leads to

$$\underline{\Pi}_V^{e,m}(\underline{r}) = \iiint_{V_1} G(\underline{r}-\underline{r}')\underline{J}_V^{e,m}(\underline{r}')dV(\underline{r}') \quad \text{with } \underline{r} \in \mathbb{R}^3 \tag{6.70}$$

and

$$\underline{\Pi}_S^{e,m}(\underline{r}) = \iint_S G(\underline{r}-\underline{r}')\underline{J}_S^{e,m}(\underline{r}')dA(\underline{r}') \text{ with } \underline{r} \in \mathbb{R}^3. \tag{6.71}$$

On account of (6.69), $G(\underline{r}-\underline{r}')$ is given by

$$G(\underline{r}-\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp[i\{(\sigma - i\omega\varepsilon)i\omega\mu\}^{\frac{1}{2}}|\underline{r}-\underline{r}'|]$$

$$\text{for all } \underline{r} \neq \underline{r}', \tag{6.72}$$

in which

$$|\underline{r}-\underline{r}'| = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}} \tag{6.73}$$

with $[\ldots]^{\frac{1}{2}} \geq 0$.

Finally, the factors $\underline{ik}$ in the right-hand sides of (6.66) and (6.67) have to be handled. In this respect we refer to (6.A5) and observe that upon Fourier inversion $\underline{ik}$ corresponds with the operator $\underline{\nabla}$ in case the relevant Fourier transform is extended over all space and on the condition that the contribution from the "sphere at infinity" vanishes. As we have seen, this situation applies to G; as (6.70) and (6.71) show, it also applies to $\underline{\Pi}^{e,m}_{V,S}$. On account of (6.A3), Fourier inversion of (6.66) and (6.67) then leads to

$$i\omega\mu[\underline{\Pi}^{e}_{V}(\underline{r}) + \underline{\Pi}^{e}_{S}(\underline{r})] + (\sigma - i\omega\varepsilon)^{-1}\underline{\nabla}\{\underline{\nabla}\cdot[\underline{\Pi}^{e}_{V}(\underline{r}) + \underline{\Pi}^{e}_{S}(\underline{r})]\}$$

$$- \underline{\nabla} \times [\underline{\Pi}^{m}_{V}(\underline{r}) + \underline{\Pi}^{m}_{S}(\underline{r})] = \{1,\ \tfrac{1}{2},\ 0\}\ \underline{E}(\underline{r})$$

$$\text{when } \underline{r} \in \{V_1,\ S,\ V_0\} \qquad (6.74)$$

and

$$-(\sigma - i\omega\varepsilon)[\underline{\Pi}^{m}_{V}(\underline{r}) + \underline{\Pi}^{m}_{S}(\underline{r})] - (i\omega\mu)^{-1}\underline{\nabla}\{\underline{\nabla}\cdot[\underline{\Pi}^{m}_{V}(\underline{r}) + \underline{\Pi}^{m}_{S}(\underline{r})]\}$$

$$+ \underline{\nabla} \times [\underline{\Pi}^{e}_{V}(\underline{r}) + \underline{\Pi}^{e}_{S}(\underline{r})] = \{1,\ \tfrac{1}{2},\ 0\}\ \underline{H}(\underline{r})$$

$$\text{when } \underline{r} \in \{V_1,\ S,\ V_0\}, \qquad (6.75)$$

in which (6.70) and (6.71) are to be substituted.

When $\underline{r} \in V_1$, the left-hand side of (6.74) is an integral representation for the electric-field strength and the left-hand side of (6.75) an integral representation for the magnetic-field strength. Obviously, integral representations for $\underline{E}(\underline{r})$ and $\underline{H}(\underline{r})$ for all $\underline{r} \in \mathbb{R}^3$ result when we replace V_1 in (6.70) by $\mathbb{R}^3$ and omit $\underline{\Pi}^{e}_{S}$ and $\underline{\Pi}^{m}_{S}$ in the left-hand sides of (6.74) and (6.75). In view of the properties of G (cf.(6.69)), the resulting electromagnetic field consists of a superposition of spherical waves that expand from each elementary volume source.

In order to save some clerical work when the expressions (6.74) and (6.75) are used on a number of occasions, we devise a notation which is somewhere in between an abstract operator notation and the long-hand notation to which (6.74) and (6.75), in conjunction with (6.70), (6.71) and (6.72) give rise. We introduce the column matrix $[\underline{J}_V]$ of the volume currents

$$[\underline{J}_V] = \begin{bmatrix} \underline{J}^e{}_V \\ \underline{J}^m{}_V \end{bmatrix}, \tag{6.76}$$

the column matrix $[\underline{J}_S]$ of the surface currents (with $\underline{n}$ chosen as in Fig. 6.5)

$$[\underline{J}_S] = \begin{bmatrix} \underline{J}^e{}_S \\ \underline{J}^m{}_S \end{bmatrix} = \begin{bmatrix} -\underline{n}\times\underline{H} \\ \underline{n}\times\underline{E} \end{bmatrix} \tag{6.77}$$

and the column matrix $\underline{F}$ of the two field quantities $\underline{E}$ and $\underline{H}$:

$$[\underline{F}] = \begin{bmatrix} \underline{E} \\ \underline{H} \end{bmatrix}. \tag{6.78}$$

Note that the elements of the matrices (6.76), (6.77) and (6.78) are vector quantities in $\mathbb{R}^3$. Further we introduce the square matrix $[\underline{\underline{\Gamma}}]$ of the electromagnetic Green's tensors

$$[\underline{\underline{\Gamma}}] = \begin{bmatrix} \underline{\underline{\Gamma}}^{ee} & \underline{\underline{\Gamma}}^{em} \\ \underline{\underline{\Gamma}}^{me} & \underline{\underline{\Gamma}}^{mm} \end{bmatrix}, \tag{6.79}$$

whose elements are defined through the following relations

$$\underline{\underline{\Gamma}}^{ee}(\underline{r}-\underline{r}')\cdot\underline{J}^e(\underline{r}') = i\omega\mu\, G(\underline{r}-\underline{r}')\underline{J}^e(\underline{r}') + (\sigma - i\omega\varepsilon)^{-1}\underline{\nabla}\{\underline{\nabla}\cdot[G(\underline{r}-\underline{r}')\underline{J}^e(\underline{r}')]\}, \tag{6.80a}$$

$$\underline{\underline{\Gamma}}^{em}(\underline{r}-\underline{r}')\cdot\underline{J}^m(\underline{r}') = -\underline{\nabla}\times[G(\underline{r}-\underline{r}')\underline{J}^m(\underline{r}')], \tag{6.80b}$$

$$\underline{\underline{\Gamma}}^{me}(\underline{r}-\underline{r}')\cdot\underline{J}^e(\underline{r}') = \underline{\nabla}\times[G(\underline{r}-\underline{r}')\underline{J}^e(\underline{r}')], \tag{6.80c}$$

$$\underline{\underline{\Gamma}}^{mm}(\underline{r}-\underline{r}')\cdot\underline{J}^m(\underline{r}') = -(\sigma - i\omega\varepsilon)G(\underline{r}-\underline{r}')\underline{J}^m(\underline{r}') - (i\omega\mu)^{-1}\underline{\nabla}\{\underline{\nabla}\cdot[G(\underline{r}-\underline{r}')\underline{J}^m(\underline{r}')]\}. \tag{6.80d}$$

Note that the elements of (6.79) are tensors of rank two in $\mathbb{R}^3$. With the aid of (6.76)-(6.80) we can rewrite (6.74) and (6.75) as the single matrix equation

$$\iiint_{V_1} [\underline{\underline{\Gamma}}(\underline{r}-\underline{r}')]\cdot[\underline{J}_V(\underline{r}')]dV(\underline{r}') + \iint_S [\underline{\underline{\Gamma}}(\underline{r}-\underline{r}')]\cdot[\underline{J}_S(\underline{r}')]dA(\underline{r}')$$

$$= \{1, \tfrac{1}{2}, 0\}[\underline{F}(\underline{r})] \quad \text{when } \underline{r} \in \{V_1, S, V_0\}. \tag{6.81}$$

Notationally, (6.81) still enables us to indicate clearly to which domain and to which field (incident, scattered or total field) the equation is applied, at the same time having reached a certain degree of compactness.

In several cases integral representations for $\underline{E}(\underline{r})$ and $\underline{H}(\underline{r})$ are needed when $\underline{r} \in V_0$. These follow from applying the procedure outlined above to the domain V_0. It is understood that in this case the contribution from the "sphere at infinity" to the surface integrals vanishes. Also, one should take care in handling expressions with $\underline{n}$ and either let $\underline{n}$ point away from V_0 or change the sign in the expressions for $\underline{\Pi}^e_S$ and $\underline{\Pi}^m_S$.

EXERCISES

Exercise 6.24. Let the sinusoidally in time varying scalar wave function $\mathrm{Re}[U(\underline{r})\exp(-i\omega t)]$ be defined in the configuration shown in Fig.6.5 and assume that U satisfies the inhomogeneous Helmholtz equation $(\underline{\nabla}\cdot\underline{\nabla} + \omega^2/c^2)U = -Q_V$ when $\underline{r} \in V_1$. Apply the spatial Fourier transform extended over V_1 to this equation and determine the resulting equation for $\tilde{U}$.

Answer: $(\underline{k}\cdot\underline{k} - \omega^2/c^2)\tilde{U} = \tilde{Q}_V + \tilde{Q}_S + i\underline{k}\cdot\underline{\tilde{P}}_S$, in which $\tilde{Q}_S = \iint_S[\underline{n}(\underline{r})\cdot\underline{\nabla}U(\underline{r})]\exp(-i\underline{k}\cdot\underline{r})\,dA(\underline{r})$ and $\underline{\tilde{P}}_S = \iint_S \underline{n}(\underline{r})U(\underline{r})\exp(-i\underline{k}\cdot\underline{r})\,dA(\underline{r})$.

Exercise 6.25. Obtain the integral relation pertaining to U defined in Exercise 6.24, by applying Fourier inversion to the expression $\tilde{U} = \tilde{\Phi}_V + \tilde{\Phi}_S + i\underline{k}\cdot\underline{\tilde{\Psi}}_S$, where $\tilde{\Phi}_V = \tilde{G}\tilde{Q}_V$, $\tilde{\Phi}_S = \tilde{G}\tilde{Q}_S$ and $\underline{\tilde{\Psi}}_S = \tilde{G}\underline{\tilde{P}}_S$ with $\tilde{G} = (\underline{k}\cdot\underline{k} - \omega^2/c^2)^{-1}$.

Answer: $\Phi_V(\underline{r}) + \Phi_S(\underline{r}) + \underline{\nabla}\cdot\underline{\Psi}_S(\underline{r}) = \{1, \tfrac{1}{2}, 0\}\,U(\underline{r})$ when $\underline{r} \in \{V_1, S, V_0\}$, in which $\Phi_V(\underline{r}) = \iiint_{V_1} G(\underline{r}-\underline{r}')\,Q_V(\underline{r}')\,dV(\underline{r}')$, $\Phi_S(\underline{r}) = \iint_S G(\underline{r}-\underline{r}')[\underline{n}(\underline{r}')\cdot\underline{\nabla}'U(\underline{r}')]\,dA(\underline{r}')$ and $\underline{\Psi}_S(\underline{r}) = \iint_S G(\underline{r}-\underline{r}')\underline{n}(\underline{r}')U(\underline{r}')\,dA(\underline{r}')$, with $G = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp[i(\omega/c)|\underline{r}-\underline{r}'|]$.

6.6 Integral relations for the scattered field

The theory developed in Section 6.5 will now be employed to derive suitable integral relations for the scattered field. By virtue of (6.14), Table 6.1 and (6.25), $\underline{E}^s$ and $\underline{H}^s$ satisfy the electromagnetic-field equations

$$\underline{\nabla} \times \underline{H}^s + i\omega\varepsilon_0\underline{E}^s = \underline{0} \quad \text{when } \underline{r} \in V_0, \tag{6.82a}$$

$$\underline{\nabla} \times \underline{E}^s - i\omega\mu_0\underline{H}^s = \underline{0} \quad \text{when } \underline{r} \in V_0, \tag{6.82b}$$

while by virtue of (6.14), Table 6.1, (6.27) and (6.28) we have, for penetrable objects,

$$\underline{\nabla} \times \underline{H}^s + i\omega\varepsilon_0\underline{E}^s = \underline{J}_1 - i\omega\underline{P}_1 \quad \text{when } \underline{r} \in V_1, \tag{6.83a}$$

$$\underline{\nabla} \times \underline{E}^s - i\omega\mu_0\underline{H}^s = i\omega\mu_0\underline{M}_1 \quad \text{when } \underline{r} \in V_1. \tag{6.83b}$$

Equation (6.83) has the advantage that the structure of its left-hand side is the same as that of (6.82).

With reference to the constitutive relations (6.57), the left-hand side of (6.53) reduces to the common left-hand sides of (6.82) and (6.83) provided that σ approaches zero through positive values, ε is replaced by ε_0 and μ by μ_0. After these changes in the coefficients have been made, we first apply (6.81) to the domain V_0 and to the scattered field. In doing so, we shall omit the contribution from the "sphere at infinity" to the surface integral by considering the lossless medium present in V_0 to be the limiting case of a dissipative medium with non-vanishing losses. (If this procedure is not followed, the case $\sigma = 0$ requires an extra limiting condition to be imposed on $\underline{E}^s$ and $\underline{H}^s$ as $r \to \infty$; this condition is called the radiation condition [6.5].) On account of (6.82), (6.81) then leads to

$$-\iint_S [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}^s{}_S(\underline{r}')]\, dA(\underline{r}')$$

$$= \{1,\ \tfrac{1}{2},\ 0\}\ [\underline{F}^s(\underline{r})] \quad \text{when } \underline{r} \in \{V_0,\ S,\ V_1\}, \tag{6.84}$$

in which $[\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]$ is obtained by replacing $G(\underline{r}-\underline{r}')$ in (6.80) by

$$G_0(\underline{r}-\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp[i\omega(\varepsilon_0\mu_0)^{\frac{1}{2}}|\underline{r}-\underline{r}'|], \tag{6.85}$$

while (cf.(6.77))

$$[\underline{J}^s{}_S] = \begin{bmatrix} -\underline{n}\times\underline{H}^s \\ \underline{n}\times\underline{E}^s \end{bmatrix} \tag{6.86}$$

and (cf.(6.78))

$$[\underline{F}^s] = \begin{bmatrix} \underline{E}^s \\ \underline{H}^s \end{bmatrix} \tag{6.87}$$

Amongst others, (6.84) expresses the scattered field in V_0 in terms of the limiting values of the tangential parts of the electric- and the magnetic-field strengths of the scattered field on S.

Next, we apply (6.81) to the domain V_1 and the incident field. By virtue of (6.14), Table 6.1, (6.23) and (6.24) we have

$$\underline{\nabla} \times \underline{H}^i + i\omega\varepsilon_0\underline{E}^i = \underline{0} \quad \text{when } \underline{r} \in V_1, \tag{6.88a}$$

$$\underline{\nabla} \times \underline{E}^i - i\omega\mu_0\underline{H}^i = \underline{0} \quad \text{when } \underline{r} \in V_1. \tag{6.88b}$$

Equation (6.81) then leads to

$$\iint_S [\underline{\Gamma}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}^i_S(\underline{r}')]\, dA(\underline{r}')$$

$$= \{1, \tfrac{1}{2}, 0\}\ [\underline{F}^i(\underline{r})] \quad \text{when } \underline{r} \in \{V_1, S, V_0\}, \tag{6.89}$$

in which

$$[\underline{J}^i_S] = \begin{bmatrix} -\underline{n}\times\underline{H}^i \\ \underline{n}\times\underline{E}^i \end{bmatrix} \tag{6.90}$$

and

$$[\underline{F}^i] = \begin{bmatrix} \underline{E}^i \\ \underline{H}^i \end{bmatrix}. \tag{6.91}$$

Subtraction of (6.89) from (6.84) yields

$$-\iint_S [\underline{\Gamma}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}_{S,0}(\underline{r}')]\, dA(\underline{r}')$$

$$= \{[\underline{F}^s(\underline{r})],\ \tfrac{1}{2}[\underline{F}_0(\underline{r})] - [\underline{F}^i(\underline{r})],\ -[\underline{F}^i(\underline{r})]\} \quad \text{when } \underline{r} \in \{V_0, S, V_1\}, \tag{6.92}$$

in which

$$[\underline{J}_{S,0}] = \begin{bmatrix} -\underline{n}\times\underline{H}_0 \\ \underline{n}\times\underline{E}_0 \end{bmatrix}. \tag{6.93}$$

When $\underline{r} \in V_0$, (6.92) (as well as (6.84)) can be used as an integral representation for the scattered field, but now the tangential parts of the electric- and the magnetic-field strengths of the total field upon approaching S via V_0 occur in the integral over S. Especially when explicit boundary conditions are known to hold on S, (6.92) may have certain advantages over (6.84).

For scattering objects that are penetrable, one can also apply (6.81) to the scattered field in V_1 and use (6.83). When the resulting expression is added to (6.84), the surface integrals cancel by virtue of the continuity of $\underline{n} \times \underline{E}^s$ and $\underline{n} \times \underline{H}^s$ across S (this is a consequence of (6.15) together with the continuity of $\underline{n} \times \underline{E}^i$ and $\underline{n} \times \underline{H}^i$ across S) and we are left with

$$\iiint_{V_1} [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}^s{}_V(\underline{r}')]\, dV(\underline{r}') = \{[\underline{F}^s(\underline{r})],\ \tfrac{1}{2}[\underline{F}_1(\underline{r})] + \tfrac{1}{2}[\underline{F}_0(\underline{r})] - [\underline{F}^i(\underline{r})],\ [\underline{F}^s(\underline{r})]\} \quad \text{when } \underline{r} \in \{V_1,\ S,\ V_0\}, \tag{6.94}$$

in which

$$[\underline{J}_V{}^s] = \begin{bmatrix} \underline{J}_1 - i\omega\underline{P}_1 \\ -i\omega\mu_0\underline{M}_1 \end{bmatrix}. \tag{6.95}$$

Equation (6.94) expresses the scattered field at any point in space not on S in terms of the contrast quantities that make the scattering object differ from its surroundings.

Representations for the spherical-wave amplitudes of the scattered field in the far-field region

In particular, (6.84), (6.92) and (6.94) can be used to obtain representations for the angularly dependent spherical-wave amplitudes of the scattered field in the far-field region. In the surface integrals over S or the volume integrals over V_1 we then let $|\underline{r}| \to \infty$ and use the results (6.B8), (6.B14) and (6.B15) of Appendix B. Application of this procedure to (6.84) and comparison of the resulting expressions with (6.39) yields

$$[\underline{f}^s(\underline{\theta})] = -[\underline{\underline{\Upsilon}}_0(\underline{\theta})]\cdot[\underline{\tilde{J}}^s{}_S(k_0\underline{\theta})] \quad \text{with } \underline{\theta} \in \Omega, \tag{6.96}$$

where

$$[\underline{f}^s] = \begin{bmatrix} \underline{e}^s \\ \underline{h}^s \end{bmatrix} \tag{6.97}$$

and

$$[\underline{\underline{Y}}_0] = \begin{bmatrix} \underline{\underline{Y}}_0^{ee} & \underline{\underline{Y}}_0^{em} \\ \underline{\underline{Y}}_0^{me} & \underline{\underline{Y}}_0^{mm} \end{bmatrix}. \tag{6.98}$$

The elements of (6.98) follow from the relations (cf.(6.80))

$$\underline{\underline{Y}}_0^{ee}\cdot\underline{\tilde{J}}^e = i\omega\mu_0[\underline{\tilde{J}}^e - \underline{\theta}(\underline{\theta}\cdot\underline{\tilde{J}}^e)], \tag{6.99a}$$

$$\underline{\underline{Y}}_0^{em}\cdot\underline{\tilde{J}}^m = -ik_0\underline{\theta} \times \underline{\tilde{J}}^m, \tag{6.99b}$$

$$\underline{\underline{Y}}_0^{me}\cdot\underline{\tilde{J}}^e = ik_0\underline{\theta} \times \underline{\tilde{J}}^e, \tag{6.99c}$$

$$\underline{\underline{Y}}_0^{mm}\cdot\underline{\tilde{J}}^m = i\omega\varepsilon_0[\underline{\tilde{J}}^m - \underline{\theta}(\underline{\theta}\cdot\underline{\tilde{J}}^m)]. \tag{6.99d}$$

In a similar way (6.92) leads to

$$[\underline{f}^s(\underline{\theta})] = -[\underline{\underline{Y}}_0(\underline{\theta})]\cdot[\underline{\tilde{J}}_{S,0}(k_0\underline{\theta})] \quad \text{with } \underline{\theta} \in \Omega \tag{6.100}$$

and (6.94) to

$$[\underline{f}^s(\underline{\theta})] = [\underline{\underline{Y}}_0(\underline{\theta})]\cdot[\underline{\tilde{J}}^s{}_V(k_0\underline{\theta})] \quad \text{with } \underline{\theta} \in \Omega. \tag{6.101}$$

Because of the structure of $[\underline{\underline{Y}}_0]$, the quantities $\underline{e}^s$ and $\underline{h}^s$ that are determined from (6.96), (6.100) or (6.101) satisfy the relations (6.40) (cf. Exercise 6.26).

EXERCISES

Exercise 6.26. Show with the aid of Eq.(6.99) that the following relations hold: $Z_0\underline{\theta} \times (\underline{\underline{Y}}_0^{me}\cdot\underline{\tilde{J}}^e) = -\underline{\underline{Y}}_0^{ee}\cdot\underline{\tilde{J}}^e$, $Z_0\underline{\theta} \times (\underline{\underline{Y}}_0^{mm}\cdot\underline{\tilde{J}}^m) = -\underline{\underline{Y}}_0^{em}\cdot\underline{\tilde{J}}^m$, $Y_0\underline{\theta} \times (\underline{\underline{Y}}_0^{ee}\cdot\underline{\tilde{J}}^e) = \underline{\underline{Y}}_0^{me}\cdot\underline{J}^e$, $Y_0\underline{\theta}\times (\underline{\underline{Y}}_0^{em}\cdot\underline{\tilde{J}}^m) = \underline{\underline{Y}}_0^{mm}\cdot\underline{\tilde{J}}^m$, $\underline{\theta}\cdot(\underline{\underline{Y}}_0^{ee}\cdot\underline{\tilde{J}}^e) = 0$, $\underline{\theta}\cdot(\underline{\underline{Y}}_0^{em}\cdot\underline{\tilde{J}}^m) = 0$, $\underline{\theta}\cdot(\underline{\underline{Y}}_0^{me}\cdot\underline{\tilde{J}}^e) = 0$, $\underline{\theta}\cdot(\underline{\underline{Y}}_0^{mm}\cdot\underline{\tilde{J}}^m) = 0$.

6.7 Reciprocity properties of the spherical-wave amplitudes of the scattered field in the far-field region for plane-wave scattering

In this section we investigate the reciprocity properties of the spherical-wave amplitudes of the scattered field in the far-field region in case the incident field is a uniform plane wave. In the frequency-domain reciprocity relation two different electromagnetic states with the same angular frequency occur. Let the superscripts A and B indicate the electromagnetic field quantities in the two states, then the electromagnetic field reciprocity relation (also called Lorentz's theorem) states that for any bounded domain V with boundary surface S we have

$$\iint_S \underline{n}\cdot(\underline{E}^A \times \underline{H}^B)\, dA = \iint_S \underline{n}\cdot(\underline{E}^B \times \underline{H}^A)\, dA, \tag{6.102}$$

provided that no sources are present in V (in which case (6.14) applies) and that

$$\underline{E}^A\cdot\underline{J}^B - i\omega\underline{E}^A\cdot\underline{D}^B - i\omega\underline{H}^B\cdot\underline{B}^A = \underline{E}^B\cdot\underline{J}^A - i\omega\underline{E}^B\cdot\underline{D}^A - i\omega\underline{H}^A\cdot\underline{B}^B$$

$$\text{at all } \underline{r} \in V. \tag{6.103}$$

We first apply (6.102) to the scattered fields and to the domain bounded internally by S of Fig.6.1 and externally by the sphere

$$S_\Delta = \{\underline{r}\,|\,\underline{r} \in \mathbb{R}^3,\ \underline{r}\cdot\underline{r} = \Delta^2\}. \tag{6.104}$$

As in this situation (6.103) is obviously satisfied (cf.(6.25)), we obtain

$$\iint_S \underline{n}\cdot(\underline{E}^{s,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{s,A})\, dA$$

$$= \iint_{S_\Delta} \underline{\theta}\cdot(\underline{E}^{s,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{s,A})\, dA. \tag{6.105}$$

However, on account of (6.39) we have

$$\iint_{S_\Delta} \underline{\theta}\cdot(\underline{E}^{s,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{s,A})\, dA$$

$$= (4\pi)^{-2}\exp(2ik_0\Delta)\iint_\Omega \underline{\theta}\cdot(\underline{e}^{s,A} \times \underline{h}^{s,B} - \underline{e}^{s,B} \times \underline{h}^{s,A})\, d\Omega$$

$$+ \text{ vanishing terms as } \Delta \to \infty. \tag{6.106}$$

In view of (6.40), however

$$\underline{\theta}\cdot(\underline{e}^{s,A} \times \underline{h}^{s,B}) = \underline{\theta}\cdot(\underline{e}^{s,B} \times \underline{h}^{s,A}) \tag{6.107}$$

and hence, (6.106) leads to

$$\lim_{\Delta\to\infty} \iint_{S_\Delta} \underline{\theta}\cdot(\underline{E}^{s,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{s,A})\, dA = 0. \tag{6.108}$$

Now, the left-hand side of (6.105) is independent of Δ. This fact, combined with (6.108), yields

$$\iint_S \underline{n}\cdot(\underline{E}^{s,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{s,A})\, dA = 0. \tag{6.109}$$

Secondly, we apply (6.102) to the incident fields and to the domain V_1 of Fig.6.1. In view of the definition of incident field, (6.103) is again satisfied and we obtain

$$\iint_S \underline{n}\cdot(\underline{E}^{i,A} \times \underline{H}^{i,B} - \underline{E}^{i,B} \times \underline{H}^{i,A})\, dA = 0 \tag{6.110}$$

Using the results of Table 6.1 it then follows from (6.109) and (6.110) that

$$\iint_S \underline{n}\cdot(\underline{E}^A_0 \times \underline{H}^B_0 - \underline{E}^B_0 \times \underline{H}^A_0)\, dA = \iint_S \underline{n}\cdot(\underline{E}^{i,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{i,A})dA$$

$$+ \iint_S \underline{n}\cdot(\underline{E}^{s,A} \times \underline{H}^{i,B} - \underline{E}^{i,B} \times \underline{H}^{s,A})\, dA. \tag{6.111}$$

Next, (6.102) is applied to the total fields in the scattering object and to the domain V_1. We now <u>require</u> that (6.103) is satisfied throughout V_1 and then obtain

$$\iint_S \underline{n}\cdot(\underline{E}^A_1 \times \underline{H}^B_1 - \underline{E}^B_1 \times \underline{H}^A_1)\, dA = 0. \tag{6.112}$$

By virtue of the boundary conditions across S ((6.15) for a penetrable object), however, the left-hand side of (6.111) equals the left-hand side of (6.112) and hence

$$\iint_S \underline{n}\cdot(\underline{E}^{i,A} \times \underline{H}^{s,B} - \underline{E}^{s,B} \times \underline{H}^{i,A})\, dA$$

$$= \iint_S \underline{n}\cdot(\underline{E}^{i,B} \times \underline{H}^{s,A} - \underline{E}^{s,A} \times \underline{H}^{i,B})\, dA. \tag{6.113}$$

For a perfectly conducting object or a perfectly conducting screen, (6.113) directly follows from (6.111) by invoking the boundary conditions (6.16) and (6.19), respectively. Note that (6.113) holds for arbitrary incident fields.

Uniform plane waves as incident fields

We now take the incident fields to be the uniform plane waves (cf.Section 6.3.)

$$\{\underline{E}^{i,A}, \underline{H}^{i,A}\} = \{\underline{e}^{i}_{\underline{\alpha}}, \underline{h}^{i}_{\underline{\alpha}}\}\exp(ik_0\underline{\alpha}\cdot\underline{r}) \tag{6.114}$$

and

$$\{\underline{E}^{i,B}, \underline{H}^{i,B}\} = \{\underline{e}^{i}_{\underline{\beta}}, \underline{h}^{i}_{\underline{\beta}}\}\exp(ik_0\underline{\beta}\cdot\underline{r}), \tag{6.115}$$

propagating in the directions of the unit vectors $\underline{\alpha}$ and $\underline{\beta}$, respectively. For this kind of excitation of the scattering object, we henceforth write $\{\underline{E}^{s}_{\underline{\alpha}}, \underline{H}^{s}_{\underline{\alpha}}\}$ in stead of $\{\underline{E}^{s,A}, \underline{H}^{s,A}\}$ and $\{\underline{E}^{s}_{\underline{\beta}}, \underline{H}^{s}_{\underline{\beta}}\}$in stead of $\{\underline{E}^{s,B}, \underline{H}^{s,B}\}$. Substituting (6.114) in the left-hand side of (6.113), using (6.32) and comparing the result with (6.96), we observe that

$$\iint_S \underline{n}\cdot(\underline{e}^{i}_{\underline{\alpha}} \times \underline{H}^{s}_{\underline{\beta}} - \underline{E}^{s}_{\underline{\beta}} \times \underline{h}^{i}_{\underline{\alpha}})\exp(ik_0\underline{\alpha}\cdot\underline{r})dA$$

$$= -(i\omega\mu_0)^{-1}\underline{e}^{i}_{\underline{\alpha}}\cdot\underline{e}^{s}_{\underline{\beta}}(-\underline{\alpha}) = (i\omega\varepsilon_0)^{-1}\underline{h}^{i}_{\underline{\alpha}}\cdot\underline{h}^{s}_{\underline{\beta}}(-\underline{\alpha}). \tag{6.116}$$

Similarly, substituting (6.115) in the right-hand side of (6.113), using (6.32) and comparing the result with (6.96), we observe that

$$\iint_S \underline{n}\cdot(\underline{e}^{i}_{\underline{\beta}} \times \underline{H}^{s}_{\underline{\alpha}} - \underline{E}^{s}_{\underline{\alpha}} \times \underline{h}^{i}_{\underline{\beta}})\exp(ik_0\underline{\beta}\cdot\underline{r})\, dA$$

$$= -(i\omega\mu_0)^{-1}\underline{e}^{i}_{\underline{\beta}}\cdot\underline{e}^{s}_{\underline{\alpha}}(-\underline{\beta}) = (i\omega\varepsilon_0)^{-1}\underline{h}^{i}_{\underline{\beta}}\cdot\underline{h}^{s}_{\underline{\alpha}}(-\underline{\beta}). \tag{6.117}$$

Equations (6.113), (6.116) and (6.117) lead to the final results

$$\underline{e}^{i}_{\underline{\alpha}}\cdot\underline{e}^{s}_{\underline{\beta}}(-\underline{\alpha}) = \underline{e}^{i}_{\underline{\beta}}\cdot\underline{e}^{s}_{\underline{\alpha}}(-\underline{\beta}) \tag{6.118}$$

and

$$\underline{h}^i_{\underline{\alpha}} \cdot \underline{h}^s_{\underline{\beta}}(-\underline{\alpha}) = \underline{h}^i_{\underline{\beta}} \cdot \underline{h}^s_{\underline{\alpha}}(-\underline{\beta}). \tag{6.119}$$

Equations (6.118) and (6.119) constitute the reciprocity relations for plane-wave scattering [6.6]. Fig.6.6 illustrates the interrelation between the different directions in these reciprocity relations.

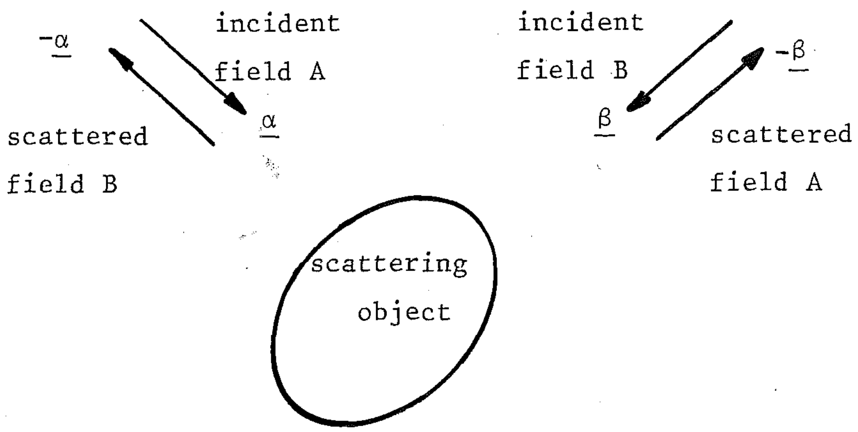

Fig.6.6. Interrelation between the different directions in the reciprocity relation for plane-wave scattering.

EXERCISES

Exercise 6.27. If the electromagnetic properties of a medium are such that Eq. (6.103) holds when the medium in State A is identical to the medium in State B, this medium is called electromagnetically reciprocal. Show that a medium whose constitutive relations are $\underline{J} = \sigma\underline{E}$, $\underline{D} = \varepsilon\underline{E}$ and $\underline{B} = \mu\underline{H}$ is electromagnetically reciprocal.

Exercise 6.28. Verify Eqs.(6.116) and (6.117).

6.8 The extinction cross-section theorem (optical theorem)

The extinction cross-section theorem or "optical theorem" relates, for plane-wave scattering, the time-averaged power both absorbed and scattered by a scattering object to the spherical-wave amplitude of the scattered field in the far-field region, observed in the forward direction. The time-averaged <u>power absorbed</u> by the scattering object is given by (note that $\underline{n}$ points away from V_1 in Fig.6.1)

$$\langle P^a \rangle_T = -\tfrac{1}{2}\,\mathrm{Re}\left[\iint_S \underline{n}\cdot(\underline{E}_0 \times \underline{H}_0^*)\, dA\right]. \tag{6.120}$$

In the right-hand side of (6.120) we use the results of Table 6.1 and obtain

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\tfrac{1}{2}\,\mathrm{Re}\left[\iint_S \underline{n}\cdot(\underline{E}^i \times \underline{H}^{i*})\, dA\right]$$

$$-\tfrac{1}{2}\,\mathrm{Re}\left[\iint_S \underline{n}\cdot(\underline{E}^{i*} \times \underline{H}^s + \underline{E}^s \times \underline{H}^{i*})\, dA\right], \tag{6.121}$$

in which (6.48) has been used and where we have used the property that the real part of a complex quantity is equal to the real part of its complex conjugate. Since the medium in which the incident field has been generated is assumed to be lossless, we have

$$\tfrac{1}{2}\,\mathrm{Re}\left[\iint_S \underline{n}\cdot(\underline{E}^i \times \underline{H}^{i*})dA\right] = 0 \tag{6.122}$$

as the incident field has no sources in V_1. Substitution of (6.122) in (6.121) yields

$$\langle P^a \rangle_T + \langle P^s \rangle_T = -\tfrac{1}{2}\,\mathrm{Re}\left[\iint_S \underline{n}\cdot(\underline{E}^{i*} \times \underline{H}^s + \underline{E}^s \times \underline{H}^{i*})\, dA\right]. \tag{6.123}$$

Equation (6.123) holds for an arbitrary incident field.

Uniform plane wave as incident field

We now take the incident field to be the uniform plane wave (cf.Section 6.3)

$$\{\underline{E}^i, \underline{H}^i\} = \{\underline{e}^i_{\alpha}, \underline{h}^i_{\alpha}\}\exp(ik_0\underline{\alpha}\cdot\underline{r}), \tag{6.124}$$

propagating in the direction of the unit vector $\underline{\alpha}$. For this kind of excitation of the scattering object we henceforth write $\{\underline{E}^s_{\alpha}, \underline{H}^s_{\alpha}\}$ in stead of $\{\underline{E}^s, \underline{H}^s\}$. Substituting (6.124) in the right-hand side of (6.123), using

(6.32) and comparing the result with (6.96), we observe that

$$\iint_S \underline{n}\cdot(\underline{e}^{i\,*}_{\underline{\alpha}} \times \underline{H}^{s}_{\underline{\alpha}} + \underline{E}^{s}_{\underline{\alpha}} \times \underline{h}^{i\,*}_{\underline{\alpha}})\exp(-ik_0\underline{\alpha}\cdot\underline{r})dA$$

$$= -(i\omega\mu_0)^{-1}\underline{e}^{i\,*}_{\underline{\alpha}}\cdot\underline{e}^{s}_{\underline{\alpha}}(\underline{\alpha}) = -(i\omega\varepsilon_0)^{-1}\underline{h}^{i\,*}_{\underline{\alpha}}\cdot\underline{h}^{s}_{\underline{\alpha}}(\underline{\alpha}). \tag{6.125}$$

Equations (6.123) and (6.125) lead to the result

$$\langle P^a\rangle_T + \langle P^s\rangle_T = \tfrac{1}{2}\,\mathrm{Im}[(\omega\mu_0)^{-1}\underline{e}^{i\,*}_{\underline{\alpha}}\cdot\underline{e}^{s}_{\underline{\alpha}}(\underline{\alpha})]$$

$$= \tfrac{1}{2}\,\mathrm{Im}[(\omega\varepsilon_0)^{-1}\underline{h}^{i\,*}_{\underline{\alpha}}\cdot\underline{h}^{s}_{\underline{\alpha}}(\underline{\alpha})]. \tag{6.126}$$

In (6.126) we next introduce the absorption cross-section $\sigma^a_{\underline{\alpha}}$ defined as

$$\sigma^a_{\underline{\alpha}} \overset{\text{def}}{=\!=} \langle P^a\rangle_T/\underline{\alpha}\cdot\langle\underline{S}^i_{\underline{\alpha}}\rangle_T \tag{6.127}$$

and the average scattering cross-section $\sigma^s_{\underline{\alpha}}$ given by (cf.(6.52))

$$\sigma^s_{\underline{\alpha}} = \langle P^s\rangle_T/\underline{\alpha}\cdot\langle\underline{S}^i_{\underline{\alpha}}\rangle_T. \tag{6.128}$$

Since (cf.(6.37) and (6.38))

$$\underline{\alpha}\cdot\langle\underline{S}^i_{\underline{\alpha}}\rangle_T = \tfrac{1}{2}Y_0(\underline{e}^{i}_{\underline{\alpha}}\cdot\underline{e}^{i\,*}_{\underline{\alpha}}) = \tfrac{1}{2}Z_0(\underline{h}^{i}_{\underline{\alpha}}\cdot\underline{h}^{i\,*}_{\underline{\alpha}}), \tag{6.129}$$

we finally obtain

$$\sigma^a_{\underline{\alpha}} + \sigma^s_{\underline{\alpha}} = \frac{1}{k_0}\,\frac{\mathrm{Im}[\underline{e}^{i\,*}_{\underline{\alpha}}\cdot\underline{e}^{s}_{\underline{\alpha}}(\underline{\alpha})]}{\underline{e}^{i}_{\underline{\alpha}}\cdot\underline{e}^{i\,*}_{\underline{\alpha}}} = \frac{1}{k_0}\,\frac{\mathrm{Im}[\underline{h}^{i\,*}_{\underline{\alpha}}\cdot\underline{h}^{s}_{\underline{\alpha}}(\underline{\alpha})]}{\underline{h}^{i}_{\underline{\alpha}}\cdot\underline{h}^{i\,*}_{\underline{\alpha}}}. \tag{6.130}$$

In (6.130) the sum of $\sigma^a_{\underline{\alpha}}$ and $\sigma^s_{\underline{\alpha}}$ occurs; this quantity is also known as the <u>extinction cross-section</u> of the scattering object. Therefore, (6.130) is called the extinction cross-section theorem [6.7, 6.8]. (In the quantum-mechanical theory of particle scattering the corresponding theorem is known as the "optical theorem".)

EXERCISES

<u>Exercise 6.29.</u> Verify Eq. (6.125).

<u>Exercise 6.30.</u> Show that for a perfectly conducting scattering object or for a perfectly conducting screen we have $\langle P^a\rangle_T = 0$ and hence $\sigma^a_{\underline{\alpha}} = 0$.

6.9 Integral-equation formulation of the scattering by a penetrable object

The integral-equation formulation of the scattering by a penetrable object involves the following steps to be carried out consecutively.

(1) In (6.94) the point $\underline{r}$ is chosen in V_1. Using the results of Table 6.1 in the right-hand side, the equation is rewritten as

$$\iiint_{V_1} [\underline{\Gamma}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}^s_V(\underline{r}')]\, dV(\underline{r}')$$

$$= [\underline{F}_1(\underline{r})] - [\underline{F}^i(\underline{r})] \quad \text{when } \underline{r} \in V_1, \tag{6.131}$$

in which

$$[\underline{J}^s_V] = \begin{bmatrix} \underline{J}_1 - i\omega\underline{P}_1 \\ -i\omega\mu_0\underline{M}_1 \end{bmatrix} = \begin{bmatrix} \underline{J}_1 - i\omega\underline{D}_1 + i\omega\varepsilon_0\underline{E}_1 \\ -i\omega\underline{B}_1 + i\omega\mu_0\underline{H}_1 \end{bmatrix} . \tag{6.132}$$

(2) After the constitutive matrix in (6.26) has been specified for the object under consideration, we eliminate $\underline{J}_1$, $\underline{D}_1$ and $\underline{B}_1$ from (6.26) and (6.131). A system of differential-integral equations then results with $\underline{E}_1(\underline{r})$ and $\underline{H}_1(\underline{r})$ (or $[\underline{F}_1(\underline{r})]$) with $\underline{r} \in V_1$ as unknown functions and $\underline{E}^i(\underline{r})$ and $\underline{H}^i(\underline{r})$ (or $[\underline{F}^i(\underline{r})]$) with $\underline{r} \in V_1$ as known functions.

(3) By some method, the system of differential-integral equations is solved (for the numerical techniques that are available, we refer to Section 6.13).

(4) Using the calculated values of $\underline{E}_1(\underline{r})$ and $\underline{H}_1(\underline{r})$ for $\underline{r} \in V_1$, we determine $\underline{J}_1(\underline{r})$, $\underline{D}_1(\underline{r})$ and $\underline{B}_1(\underline{r})$ for $\underline{r} \in V_1$ with the aid of (6.26) and from these values calculate $[\underline{J}^s_V]$.

(5) The calculated value of $[\underline{J}^s_V]$ is used in (6.94) to calculate the scattered field anywhere in $\mathbb{R}^3\backslash S$. When $\underline{r} \in V_0$ this involves the numerical evaluation of the volume integral over V_1; when $\underline{r} \in V_1$, the scattered field is far more easily calculated by subtracting the known values of the incident field from the calculated values of the total field.

It is observed that the system of differential-integral equations (6.131) uncouples into two separate differential-integral equations for $\underline{E}_1$ or $\underline{H}_1$, respectively, in case the scattering object shows either no magnetic or no electric contrast with its surroundings.

EXERCISES

Exercise 6.31. Express $[\underline{J}^s{}_V]$ in Eq.(6.131) in terms of $\underline{E}_1$ and $\underline{H}_1$ for a scattering object whose constitutive relations are $\underline{J}_1(\underline{r}) = \sigma_1(\underline{r})\,\underline{E}_1(\underline{r})$, $\underline{D}_1(\underline{r}) = \varepsilon_1(\underline{r})\underline{E}_1(\underline{r})$, $\underline{B}_1(\underline{r}) = \mu_1(\underline{r})\,\underline{H}_1(\underline{r})$.

Answer:

$$[\underline{J}^s{}_V(\underline{r})] = \begin{bmatrix} [\sigma_1(\underline{r}) - i\omega\{\varepsilon_1(\underline{r}) - \varepsilon_0\}]\underline{E}_1(\underline{r}) \\ \\ -i\omega\{\mu_1(\underline{r}) - \mu_0\}\underline{H}_1(\underline{r}) \end{bmatrix} .$$

6.10 Integral-equation formulation of the scattering by an electrically impenetrable (perfectly conducting) object

The integral-equation formulation of the scattering by an electrically impenetrable (perfectly conducting) object runs along the following lines.

(1) In (6.92) the point $\underline{r}$ is chosen on S. On the resulting equation we perform the operation $\underline{n}(\underline{r}) \times \ldots$ and write the result as

$$-\underline{n}(\underline{r}) \times \iint_S [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}_{S,0}(\underline{r}')]\,dA(\underline{r}')$$
$$= \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[\underline{J}_{S,0}(\underline{r})] - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}[\underline{J}^i{}_S(\underline{r})] \quad \text{when } \underline{r} \in S, \qquad (6.133)$$

in which

$$[\underline{J}_{S,0}] = \begin{bmatrix} -\underline{n} \times \underline{H}_0 \\ \underline{n} \times \underline{E}_0 \end{bmatrix} \qquad (6.134)$$

and

$$[\underline{J}^i{}_S] = \begin{bmatrix} -\underline{n} \times \underline{H}^i \\ \underline{n} \times \underline{E}^i \end{bmatrix} . \qquad (6.135)$$

(2) On account of the boundary condition (6.16), however, the magnetic surface-current density on S vanishes and the scattering object shows no magnetic contrast with its surroundings. As a consequence, the system of differential-integral equations (6.133) separates into

$$-\underline{n}(\underline{r}) \times \iint_S \underline{\underline{\Gamma}}^{ee}_0(\underline{r}-\underline{r}')\cdot\{-\underline{n}(\underline{r}') \times \underline{H}_0(\underline{r}')\}\, dA(\underline{r}')$$

$$= -\underline{n}(\underline{r}) \times \underline{E}^i(\underline{r}) \quad \text{when } \underline{r} \in S \tag{6.136}$$

and

$$-\underline{n}(\underline{r}) \times \iint_S \underline{\underline{\Gamma}}^{me}_0(\underline{r}-\underline{r}')\cdot\{-\underline{n}(\underline{r}') \times \underline{H}_0(\underline{r}')\}\, dA(\underline{r}')$$

$$= \tfrac{1}{2}\, \underline{n}(\underline{r}) \times \underline{H}_0(\underline{r}) - \underline{n}(\underline{r}) \times \underline{H}^i(\underline{r}) \quad \text{when } \underline{r} \in S, \tag{6.137}$$

respectively. Either (6.136) or (6.137) can be used to solve the scattering problem. In both equations, $\underline{J}^e_{S,0} = -\underline{n} \times \underline{H}_0$ occurs as unknown function.

(3) By some method, either (6.136) or (6.137) is solved. (For the numerical techniques that are available, we refer to Section 6.13.)

(4) The calculated value of $\underline{J}^e_{S,0}$ is used in (6.92) with $\underline{r} \in V_0$ to calculate the scattered field anywhere in the domain outside the scattering object.

EXERCISES

Exercise 6.32. Verify that $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}_S] = \begin{bmatrix} \underline{n} \times \underline{E} \\ \underline{n} \times \underline{H} \end{bmatrix}$.

6.11 Integral-equation formulation of the scattering by a homogeneous, penetrable object

In this section we discuss the scattering by a homogeneous, penetrable object whose constitutive relations are

$$\underline{J}_1 = \sigma_1 \underline{E}_1, \ \underline{D}_1 = \varepsilon_1 \underline{E}_1, \ \underline{B}_1 = \mu_1 \underline{H}_1, \tag{6.138}$$

in which σ_1, ε_1 and μ_1 are constants that may have complex values. Now, a homogeneous,penetrable scattering object is a special case of a penetrable object in general and therefore, the theory developed in Section 6.9 remains valid. In the present case, however, it is also possible to formulate the problem in terms of integral equations that hold on the boundary surface S of the scattering object. For this purpose, we need the relations that follow from (6.81) when this equation is applied to the total field in the domain V_1 occupied by the scattering object, while for the electromagnetic Green's tensors we take the ones pertaining to a medium with conductivity σ_1, permittivity ε_1 and permeability μ_1. Since in

$$\underline{\nabla} \times \underline{H}_1 - (\sigma_1 - i\omega\varepsilon_1)\, \underline{E}_1 = \underline{0}, \tag{6.139a}$$

$$\underline{\nabla} \times \underline{E}_1 - i\omega\mu_1 \underline{H}_1 = \underline{0}, \tag{6.139b}$$

no volume currents occur, (6.81) leads to

$$\iint_S [\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')]\cdot[\underline{J}_{S,1}(\underline{r}')]\, dA(\underline{r}')$$

$$= \{1, \tfrac{1}{2}, 0\}\ [\underline{F}_1(\underline{r})] \ \text{ when } \underline{r} \in \{V_1, S, V_0\}, \tag{6.140}$$

in which $[\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')]$ is obtained by replacing $G(\underline{r}-\underline{r}')$ in (6.80) by

$$G_1(\underline{r}-\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp(ik_1|\underline{r}-\underline{r}'|), \tag{6.141}$$

with

$$k_1 = \omega(\varepsilon_1\mu_1)^{\frac{1}{2}}(1 - \sigma_1/i\omega\varepsilon_1)^{\frac{1}{2}}, \tag{6.142}$$

while (cf.(6.77))

$$[\underline{J}_{S,1}] = \begin{bmatrix} -\underline{n} \times \underline{H}_1 \\ \underline{n} \times \underline{E}_1 \end{bmatrix} \tag{6.143}$$

and (cf.(6.78))

$$[\underline{F}_1] = \begin{bmatrix} \underline{E}_1 \\ \underline{H}_1 \end{bmatrix} . \tag{6.144}$$

Equation (6.140) is combined with (6.92) to lead to the desired integral--equation formulation. On account of the boundary conditions (6.15) we have $[\underline{J}_{S,0}] = [\underline{J}_{S,1}]$. To exhibit this property, we shall henceforth write

$$[\underline{J}_S] = \begin{bmatrix} -\underline{n} \times \underline{H}_0 \\ \underline{n} \times \underline{E}_0 \end{bmatrix} = \begin{bmatrix} -\underline{n} \times \underline{H}_1 \\ \underline{n} \times \underline{E}_1 \end{bmatrix} . \tag{6.145}$$

The relation between scattered field and the contrast of the scattering object with its surroundings is brought into the formulas when (6.140) is added to (6.92). This yields

$$\iint_S \{[\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')] - [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\}\cdot[\underline{J}_S(\underline{r}')]\, dA(\underline{r}')$$

$$= \{[\underline{F}^s(\underline{r})],\ \tfrac{1}{2}[\underline{F}_0(\underline{r})] + \tfrac{1}{2}[\underline{F}_1(\underline{r})] - [\underline{F}^i(\underline{r})],\ [\underline{F}^s(\underline{r})]\}$$

$$\text{when } \underline{r} \in \{V_0,\ S,\ V_1\}. \tag{6.146}$$

For the kind of scattering object under consideration, the left-hand side of (6.146) therefore yields a representation for the scattered field at any $\underline{r} \in \mathbb{R}^3\backslash S$, the contrast of the object being manifest in $[\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')] - [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]$.

The integral-equation formulation of the scattering by a homogeneous, penetrable object runs along the following lines.

(1) In (6.92) and (6.140) the point $\underline{r}$ is chosen on S. On the resulting equations we perform the operation $\underline{n}(\underline{r}) \times \ldots$ and write the result as

$$-\underline{n}(\underline{r}) \times \iint_S [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}_S(\underline{r}')]\, dA(\underline{r}')$$

$$= \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}_S(\underline{r})] - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}^i{}_S(\underline{r})] \text{ when } \underline{r} \in S \tag{6.147}$$

and

$$\underline{n}(\underline{r}) \times \iint_S [\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')]\cdot[\underline{J}_S(\underline{r}')]dA(\underline{r}')$$
$$= \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}_S(\underline{r})] \quad \text{when } \underline{r} \in S. \qquad (6.148)$$

Addition of (6.147) and (6.148) yields

$$\underline{n}(\underline{r}) \times \iint_S \{[\underline{\underline{\Gamma}}_1(\underline{r}-\underline{r}')] - [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\}\cdot[\underline{J}_S(\underline{r}')]\, dA(\underline{r}')$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}_S(\underline{r})] - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [\underline{J}^i{}_S(\underline{r})] \quad \text{when } \underline{r} \in S, \qquad (6.149)$$

which also follows from (6.146).

(2) In the integral equations (6.147), (6.148) and (6.149), $[\underline{J}_S]$ is the unknown quantity. Now, (6.147) and (6.148) together constitute a number of equations that is twice as large as the number of unknown quantities. Therefore, an appropriate number of equations is to be selected from both (6.147) and (6.148) to solve $[\underline{J}_S]$. Equation (6.149) can serve as such, but it is not the only possibility.

(3) By some method, $[\underline{J}_S]$ is calculated. (For the relevant techniques we refer to Section 6.13.)

(4) The calculated value of $[\underline{J}_S]$ is used in (6.92) with $\underline{r} \in V_0$ to calculate the scattered field anywhere in the domain outside the scattering object and in (6.140) with $\underline{r} \in V_1$ to calculate the total field anywhere in the scattering object. Also, the calculated value of $[\underline{J}_S]$ can be substituted in (6.146) to calculate the scattered field anywhere in $\mathbb{R}^3 \backslash S$.

6.12 Integral-equation formulation of the scattering by a perfectly conducting screen

When the scattering object is a screen, the domain occupied by it reduces to a lamina. The two sides of the lamina are called S^- and S^+, respectively (Fig.6.4). The left-hand side of (6.92) then reduces to a surface integral extended over S^- and S^+, while by virtue of the boundary conditions (6.19)

$$[\underline{J}_{S,0}(\underline{r}')] = \begin{bmatrix} -\underline{n}^{-}(\underline{r}') \times \underline{H}_0^{-}(\underline{r}') \\ \underline{0} \end{bmatrix} \quad \text{when } \underline{r}' \in S^{-} \tag{6.150}$$

and

$$[\underline{J}_{S,0}(\underline{r}')] = \begin{bmatrix} -\underline{n}^{+}(\underline{r}') \times \underline{H}_0^{+}(\underline{r}') \\ \underline{0} \end{bmatrix} \quad \text{when } \underline{r}' \in S^{+}. \tag{6.151}$$

For any $\underline{r} \in V_0$ we combine the two surface integrals to an integral over one of the faces of the screen and obtain

$$\iint_{S^{\pm}} \underline{\underline{\Gamma}}^{ee}_{0}(\underline{r}-\underline{r}')\cdot\underline{K}(\underline{r}')\, dA(\underline{r}')$$

$$= \underline{E}^{s}(\underline{r}) \quad \text{when } \underline{r} \in V_0 \tag{6.152}$$

and

$$\iint_{S^{\pm}} \underline{\underline{\Gamma}}^{me}_{0}(\underline{r}-\underline{r}')\cdot\underline{K}(\underline{r}')\, dA(\underline{r}')$$

$$= \underline{H}^{s}(\underline{r}) \quad \text{when } \underline{r} \in V_0, \tag{6.153}$$

in which

$$\underline{K}(\underline{r}') = \{\underline{n}(\underline{r}') \times \underline{H}_0(\underline{r}')\}_{\underline{r}'\in S^-} + \{\underline{n}(\underline{r}') \times \underline{H}_0(\underline{r}')\}_{\underline{r}'\in S^+} \tag{6.154}$$

is the equivalent volume-source electric-current density that generates the scattered field and is concentrated in the perfectly conducting lamina. The left-hand side of (6.152) has a tangential part that is continuous when $\underline{r}$ crosses the screen and hence, reusing the boundary conditions (6.19) we arrive at

$$\underline{n}^{\pm}(\underline{r}) \times \iint_{S^{\pm}} \underline{\underline{\Gamma}}^{ee}_{0}(\underline{r}-\underline{r}')\cdot\underline{K}(\underline{r}')\, dA(\underline{r}') = -\underline{n}^{\pm}(\underline{r}) \times \underline{E}^{i}(\underline{r}) \text{ when } \underline{r} \in S^{\pm}. \tag{6.155}$$

Note that (6.155) constitutes a single equation as the incident field is continuous across the screen and $\underline{n}^{+} = -\underline{n}^{-}$ (Fig.6.4). The tangential part of the left-hand side of (6.153) on the other hand is discontinuous when $\underline{r}$ crosses the screen. Its properties are such that when the operation $\underline{n}(\underline{r}) \times$ ...is performed and the two expressions for $\underline{r} \in S^{-}$ and $\underline{r} \in S^{+}$ are added, we obtain

$$\underline{K}(\underline{r}) = \{\underline{n}(\underline{r}) \times \underline{H}^{s}(\underline{r})\}_{\underline{r}\in S^-} + \{\underline{n}(\underline{r}) \times \underline{H}^{s}(\underline{r})\}_{\underline{r}\in S^+}. \qquad (6.156)$$

Again using the fact that the incident field is continuous across the screen, it follows that

$$\{\underline{n}(\underline{r}) \times \underline{H}^{s}(\underline{r})\}_{\underline{r}\in S^-} + \{\underline{n}(\underline{r}) \times \underline{H}^{s}(\underline{r})\}_{\underline{r}\in S^+}$$

$$= \{\underline{n}(\underline{r}) \times \underline{H}_0(\underline{r})\}_{\underline{r}\in S^-} + \{\underline{n}(\underline{r}) \times \underline{H}_0(\underline{r})\}_{\underline{r}\in S^+}. \qquad (6.157)$$

Equations (6.156), (6.157) and (6.154) together are consistent, but do not lead to a relation from which $\underline{K}$ can be determined. This feature is common to all scattering problems where a scattering object of vanishing thickness is involved [6.9].

The integral-equation formulation for the scattering problem under consideration then runs as follows.

(1) Equation (6.155) is used to calculate $\underline{K}$. (For the relevant techniques we refer to Section 6.13.)

(2) The calculated value of $\underline{K}$ is used in (6.152) and (6.153) to calculate the scattered field anywhere in V_0.

6.13 The method of moments

As we have seen in Sections 6.9, 6.10, 6.11 and 6.12, the major problem in the integral-equation formulation of scattering problems is to solve the relevant integral equation (Eqs. (6.131), (6.136) or (6.137), an appropriate number of equations from (6.147) and (6.148), and (6.155), respectively). Except for the very few configurations where analytical techniques are applicable, the solution of the integral-equations has to be obtained by numerical methods. In practice, the latter will have to be implemented on a high-speed digital computer. The most wide-spread technique in this respect is the method of moments ([6.10], [6.11]). To elucidate the method we select the integral equation (6.131) which is repeated here for convenience:

$$\iiint_{V_1} [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')]\cdot[\underline{J}^s{}_V(\underline{r}')]\, dV(\underline{r}')$$

$$= [\underline{F}_1(\underline{r})] - [\underline{F}^i(\underline{r})] \quad \text{when } \underline{r} \in V_1. \tag{6.158}$$

The different steps of the procedure are discussed below.

(1) A suitable, complete, sequence of expansion functions $\{[\underline{f}_n(\underline{r})]\}$ is selected to serve as a basis for the expansion of the unknown vector $[\underline{F}_1(\underline{r})]$. The vectors $[\underline{f}_n(\underline{r})]$ are defined for $\underline{r} \in V_1$. The "completeness" implies that $[\underline{F}_1(\underline{r})]$ can be represented as

$$[\underline{F}_1(\underline{r})] = \textstyle\sum_n X_n[\underline{f}_n(\underline{r})] \quad \text{when } \underline{r} \in V_1, \tag{6.159}$$

provided that the sequence of expansion coefficients $\{X_n\}$ has been chosen properly.

(2) Using (6.159) and the constitutive matrix for the scattering object under consideration, we obtain an expansion for $[\underline{J}^s{}_V]$ which we write as

$$[\underline{J}^s{}_V(\underline{r})] = \textstyle\sum_n X_n[\underline{j}_n(\underline{r})] \quad \text{when } \underline{r} \in V_1, \tag{6.160}$$

where $[\underline{j}_n(\underline{r})]$ directly follows from $[\underline{f}_n(\underline{r})]$ and the constitutive coefficients.

(3) Next, a suitable sequence of weighting functions $\{[\underline{w}_m(\underline{r})]\}$ is chosen. The vectors $[\underline{w}_m(\underline{r})]$ are defined for $\underline{r} \in V_1$. They are used to "weight" the integral equation (6.158) over the domain V_1. The weighting procedure implies that the condition for (6.158) to hold for any $\underline{r} \in V_1$ is replaced by

$$\iiint_{V_1} [\underline{w}_m(\underline{r})]\, dV(\underline{r})\cdot[\text{left-hand side of (6.158)}]$$

$$= \iiint_{V_1} [\underline{w}_m(\underline{r})]\, dV(\underline{r})\cdot[\text{right-hand side of (6.158)}] \text{ for all m.} \tag{6.161}$$

(4) The weighting procedure (6.161) and the expansions (6.159) and (6.160) are combined to lead to the following system of linear, algebraic equations

$$\sum_n A_{m,n} X_n = \sum_n B_{m,n} X_n - Y_m \quad \text{for all } m, \tag{6.162}$$

where

$$A_{m,n} = \iiint_{V_1} [\underline{w}_m(\underline{r})] \, dV(\underline{r}) \cdot \iiint_{V_1} [\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')] \cdot [\underline{j}_n(\underline{r}')] \, dV(\underline{r}'), \tag{6.163}$$

$$B_{m,n} = \iiint_{V_1} [\underline{w}_m(\underline{r})] \cdot [\underline{f}_n(\underline{r})] \, dV(\underline{r}), \tag{6.164}$$

$$Y_m = \iiint_{V_1} [\underline{w}_m(\underline{r})] \cdot [\underline{F}^i(\underline{r})] \, dV(\underline{r}). \tag{6.165}$$

Equation (6.162) replaces the integral equation (6.158). The geometry and the contrast of the scattering object with its surroundings reflect themselves in $A_{m,n}$. The coefficients $B_{m,n}$ exclusively depend on the choice of expansion and weighting functions. Finally, the coefficients Y_m characterize the way in which the scattering object is excited.

(5) The first step in the numerical procedure consists of computing $A_{m,n}$, $B_{m,n}$ and Y_m. This computation only involves numerical integrations over the domain V_1. Sometimes it is advantageous to use spatial Fourier transforms on this occasion (cf. Section 6.5). Let us write

$$[\underline{\underline{\Gamma}}_0(\underline{r}-\underline{r}')] = (2\pi)^{-3} \iiint_{\underline{k}\text{-space}} \exp\{i\underline{k}\cdot(\underline{r}-\underline{r}')\} [\underline{\underline{\tilde{\Gamma}}}_0(\underline{k})] \, dV(\underline{k}) \tag{6.166}$$

and introduce

$$[\underline{\tilde{f}}_n(\underline{k})] \stackrel{\text{def}}{=} \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r}) [\underline{f}_n(\underline{r})] \, dV(\underline{r}), \tag{6.167}$$

$$[\underline{\tilde{j}}_n(\underline{k})] \stackrel{\text{def}}{=} \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r}) [\underline{j}_n(\underline{r})] \, dV(\underline{r}), \tag{6.168}$$

$$[\underline{\tilde{w}}_m(\underline{k})] \stackrel{\text{def}}{=} \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r}) [\underline{w}_m(\underline{r})] \, dV(\underline{r}). \tag{6.169}$$

Using (6.166)-(6.169) in (6.163)-(6.164) we obtain

$$A_{m,n} = (2\pi)^{-3}\iiint_{\underline{k}\text{-space}}[\underline{\tilde{w}}_m(-\underline{k})]\cdot[\underline{\underline{\tilde{\Gamma}}}_0(\underline{k})]\cdot[\underline{\tilde{j}}_n(\underline{k})]\ dV(\underline{k}), \qquad (6.170)$$

$$B_{m,n} = (2\pi)^{-3}\iiint_{\underline{k}\text{-space}}[\underline{\tilde{w}}_m(-\underline{k})]\cdot[\underline{\tilde{f}}_n(\underline{k})]\ dV(\underline{k}). \qquad (6.171)$$

Whether or not (6.170) and (6.171) are easier to evaluate than (6.163) and (6.164) depends on the situation at hand.

(6) The second step in the numerical procedure consists of truncating the system of equations (6.162) to a square one of finite order N and solving $X_1,\ldots,X_N$ from the remaining equations by some standard subroutine. It is clear that the present step introduces an error, the magnitude of which is decisive for the final accurary of the solution of the scattering problem.

(7) The computed values of $X_1,\ldots,X_N$ are substituted in (6.159) and (6.160) which are then approximated by

$$[\underline{F}_1(\underline{r})] \approx \sum_{n=1}^{N} X_n[\underline{f}_n(\underline{r})] \quad \text{when } \underline{r} \in V_1 \qquad (6.172).$$

and

$$[\underline{J}^s{}_V(\underline{r})] \approx \sum_{n=1}^{N} X_n[\underline{j}_n(\underline{r})] \quad \text{when } \underline{r} \in V_1. \qquad (6.173)$$

(8) The computed results (6.172) and (6.173) are used to compute the scattered field anywhere in $\mathbb{R}^3\backslash S$ (cf.Section 6.9).

Error estimation

Once the computations have been performed, the crucial question as to the accuracy of the answers is raised. No detailed error estimation of the integral-equation method exists and one has to content oneself with a few checks that can be carried out. In this respect one can:

(a) increase the number of equations that is taken into account in (6.162) and see how much the values of the coefficients X_n change,

(b) verify to which accuracy the reciprocity relations (6.118) and (6.119) are satisfied,

(c) verify to which accuracy the extinction cross-section theorem (6.130) is satisfied.

Note that (a) works for any incident field, while (b) and (c) are restricted to plane waves as incident fields. In practice, (a) is an effective check; (b) and (c) are less sensitive to the inaccuracy in $\{X_n\}$.

Choice of the expansion and weighting functions

A general guideline as to the choice of the expansion functions is that they should reproduce as closely as possible the peculiarities of the field quantity that they are to represent. Known values at the boundary of the domain where the relevant integral equation holds, singular behaviour in the neighbourhood of edges, etc. should reflect themselves in each specimen of the sequence of expansion functions. From a mathematical point of view, a sequence of orthogonal functions is preferred, normalization is usually not worth the effort. As to the weighting functions, there are fewer restrictions. Mathematically it has some advantages to let the sequence of weighting functions coincide with the sequence of expansion functions. If, moreover, the two identical sequences are orthogonal, the coefficients $B_{m,n}$ (cf.(6.164)) vanish when $m \neq n$.

Since at this point it is impossible to give a review that would reasonably cover the literature on the subject, we confine ourselves to the few indications presented in Table 6.3.

Table 6.3. Expansion and weighting functions in the application of the method of moments to scattering problems

expansion or weighting function	configuration
unit pulse function $\delta(\underline{r}-\underline{r}_n)$ where $\{\underline{r}_n\}$ is a selected set of points in or on the scattering object	general Refs.[6.1], [6.11]

expansion or weighting function	configuration
rectangle function: $\text{rect}_n(\underline{r})=1$ if $\underline{r} \in \Delta V_n$, $\text{rect}_n(\underline{r})=0$ if $\underline{r} \notin \Delta V_n$, where $\{\Delta V_n\}$ is a suitable subdivision of the scattering object	general Refs.[6.1], [6.11]
linear interpolation polynomial between a selected set of points in or on the scattering object (cf.Eq.(6.7) and Exercise 6.7)	general Ref.[6.11]
eigenfunctions of Laplace's equation in elliptic-cylinder coordinates	perfectly conducting plane strip Ref.[6.12]
eigenfunctions of Laplace's equation in spheroidal coordinates	perfectly conducting circular disk Ref.[6.13]
trigonometric functions	imperfectly conducting plane strip Ref.[6.14]
periodic cubic splines	periodic boundary Ref.[6.15]

Computing time

The time consuming elements in the computation are the evaluation of the coefficients $A_{m,n}$ (either a double integral over the scattering object or an integral over the entire $\underline{k}$-space) and the computation of $X_1, \ldots, X_N$ from the system of linear, algebraic equations, especially if N has to be chosen large in order to attain to the desired accuracy. In this respect we recommend to put sufficient effort in choosing the expansion and weighting functions, since more sophistication at this stage may lead to a smaller value of N later on.

EXERCISES

Exercise 6.33. Show that for the plane wave of Eq.(6.29) as incident field the quantity Y_m given by Eq.(6.165) can be written as $Y_m = [\underline{\tilde{w}}_m(-k_0\underline{\alpha})]\cdot[\underline{f}^i_{\underline{\alpha}}]$, where $[\underline{\tilde{w}}_m]$ is given by Eq.(6.169) and $[\underline{f}^i_{\underline{\alpha}}] = \begin{bmatrix} \underline{e}^i_{\underline{\alpha}} \\ \underline{h}^i_{\underline{\alpha}} \end{bmatrix}$.

Exercise 6.34. Use Eqs.(6.101) and (6.160) to show that the expression for the spherical-wave amplitudes in the far-field region can be written as $[\underline{f}^s(\underline{\theta})] = \sum_n X_n[\underline{Y}_0(\underline{\theta})]\cdot[\underline{\tilde{j}}_n(k_0\underline{\theta})]$, where $[\underline{\tilde{j}}_n]$ is given by Eq.(6.168).

6.14 Concluding remarks

The integral-equation formulation of scattering and diffraction problems yields, in principle, accurate results for a wide variety of scattering objects as far as their shape, dimensions and physical properties are concerned. In the preceding sections we have developed the theory for objects with arbitrary geometry. If this geometry belongs to a special class, however, it will on the whole be advantageous to adapt the procedure of formulating the problem to the relevant geometry right from the beginning. Special geometries in this sense are:

(a) cylindrical objects,

(b) objects with rotational symmetry,

(c) screens in a planar configuration,

(d) object whose boundaries show spatial periodicity, e.g. optical gratings.

A detailed analysis of the relevant techniques is beyond the scope of the present chapter.

Uniqueness of the solution

Another problem of interest is the question whether the solution of the integral equations in Sections 6.9, 6.10, 6.11 and 6.12 is unique. With a few exceptions, this is indeed the case. For a perfectly conducting object (Section 6.10), the uniqueness is disturbed at a set of isolated frequencies. The latter are related to the natural frequencies of the free electromagnetic oscillations that would occur in the domain V_1 if this were filled with the surrounding medium and if appropriate boundary conditions of the homogeneous type would be invoked on S. For details in this respect we refer to the literature ([6.16], [6.17], [6.18], [6.19]).

Presentation of the results

A final remark will be made about the presentation of the results. The quantities of interest are:

(a) the field distribution inside the scattering object and on its surface,

(b) the angular distribution of the spherical-wave amplitudes of the scattered field in the far-field region,

(c) the angular distribution of the radiation intensity,

(d) the time-averaged scattered power,

(e) the time-averaged absorbed power,

and in case of plane-wave excitation

(f) the scattering cross-section,

(g) the average scattering cross-section over all directions of observation,

(h) the absorption cross-section.

All results should preferably be presented in an appropriately normalized form.

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APPENDIX

6.A The spatial Fourier transform and some of its properties

In this appendix we briefly discuss the spatial Fourier transform and its properties as far as they are needed in the analysis of scattering problems. We shall present the formulas applying to scalar functions of position; the results are easily extended to vector functions of position.

Let f be some complex- or real-valued function of position $\underline{r}$, defined on a bounded subdomain V_1 of $\mathbb{R}^3$. The boundary of V_1 is the closed surface S (Fig. 6.7). S is assumed to be sufficiently regular, i.e. the unit vector $\underline{n}$ along its outward normal is a piecewise continuous vector function of

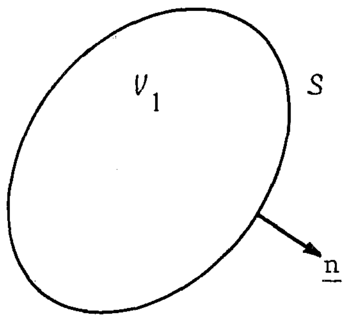

Fig. 6.7. The bounded domain V_1, interior to the closed surface S, to which the spatial Fourier transform is applied.

position. The unbounded domain exterior to S is called V_0. The spatial Fourier transform $\tilde{f}$ of f over the domain V_1 is now defined as

$$\tilde{f}(\underline{k}) \stackrel{\text{def}}{=} \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r})f(\underline{r})dV(\underline{r}) \quad \text{with } \underline{k} \in \mathbb{R}^3, \qquad (6.A1)$$

where

$$\underline{k} = k_x\underline{i}_x + k_y\underline{i}_y + k_z\underline{i}_z \qquad (6.A2)$$

denotes the wave vector. Application of the Fourier inversion theorem yields

$$(2\pi)^{-3}\iiint_{\underline{k}\text{-space}} \exp(i\underline{k}\cdot\underline{r})\tilde{f}(\underline{k})dV(\underline{k}) = \{1,\ \tfrac{1}{2},\ 0\}\ f(\underline{r})$$

$$\text{when } \underline{r} \in \{V_1,\ S,\ V_0\}, \qquad (6.A3)$$

where "$\underline{k}$-space" indicates that the integration with respect to $\underline{k}$ is to be carried out over the entire $\mathbb{R}^3$. In case $\underline{r} \in S$, the integral in the left-hand side of (6.A3) has to be interpreted as a Cauchy principal value around infinity, while the factor $\frac{1}{2}$ at the right-hand side of (6.A3) applies only to those points on S around which S is locally flat.

Next, we consider the spatial Fourier transform of $\underline{\nabla}f$. Using the definition integral (6.A1) we obtain

$$\iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r})\underline{\nabla}f(\underline{r})dV(\underline{r}) = \iiint_{V_1} \underline{\nabla}[\exp(-i\underline{k}\cdot\underline{r})f(\underline{r})]dV(\underline{r})$$

$$- \iiint_{V_1} f(\underline{r})\underline{\nabla} \exp(-i\underline{k}\cdot\underline{r})dV(\underline{r}) = \iint_S \underline{n}(\underline{r}) \exp(-i\underline{k}\cdot\underline{r})f(\underline{r})dA(\underline{r})$$

$$+ i\underline{k} \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r})f(\underline{r})dV(\underline{r}) \qquad (6.A4)$$

where we have applied Gauss' theorem to obtain the integral over S. Hence, we have

$$\widetilde{\underline{\nabla}f} = i\underline{k}\tilde{f} + \iint_S \exp(-i\underline{k}\cdot\underline{r})\underline{n}(\underline{r})f(\underline{r})\ dA(\underline{r}). \qquad (6.A5)$$

The second term at the right-hand side of (6.A5) is nothing but the spatial Fourier transform of the function $\underline{n}f$ over its domain of definition S.

In the theory we shall further encounter the product $\tilde{f}\tilde{g}$ of two spatial Fourier transforms. One of the two factors, say $\tilde{f}$, in this product can be identified with a function that is defined on a bounded subdomain V_1 of $\mathbb{R}^3$ while the other factor, $\tilde{g}$, is associated with a function that is defined in the entire $\mathbb{R}^3$. The formulas (6.A1) and (6.A3) then apply to $\tilde{f}$, while

$$\tilde{g}(\underline{k}) \stackrel{\text{def}}{=} \iiint_{\underline{r}\text{-space}} \exp(-i\underline{k}\cdot\underline{r})g(\underline{r})dV(\underline{r}) \qquad (6.A6)$$

and

$$(2\pi)^{-3}\iiint_{\underline{k}\text{-space}} \exp(i\underline{k}\cdot\underline{r})\tilde{g}(\underline{k})dV(\underline{k}) = g(\underline{r}) \quad \text{for all } \underline{r} \in \mathbb{R}^3. \qquad (6.A7)$$

Using successively (6.A1) and (6.A7) we obtain

$$(2\pi)^{-3}\iiint_{\underline{k}\text{-space}} \exp(i\underline{k}\cdot\underline{r})\tilde{f}(\underline{k})\tilde{g}(\underline{k})dV(\underline{k})$$

$$= (2\pi)^{-3}\iiint_{\underline{k}\text{-space}} \exp(i\underline{k}\cdot\underline{r})\tilde{g}(\underline{k})dV(\underline{k}) \iiint_{V_1} \exp(-i\underline{k}\cdot\underline{r}')f(\underline{r}')dV(\underline{r}')$$

$$= \iiint_{V_1} f(\underline{r}')dV(\underline{r}')(2\pi)^{-3}\iiint_{\underline{k}\text{-space}} \exp[i\underline{k}\cdot(\underline{r}-\underline{r}')]\tilde{g}(\underline{k})dV(\underline{k})$$

$$= \iiint_{V_1} f(\underline{r}')g(\underline{r}-\underline{r}')dV(\underline{r}') \quad \text{with } \underline{r} \in \mathbb{R}^3. \tag{6.A8}$$

Hence, the Fourier inversion of $\tilde{f}\tilde{g}$ is the spatial convolution of f and g. Note, that in (6.A8) we have $\underline{r} \in \mathbb{R}^3$.

6.B The spherical-wave amplitudes of the vector potentials and their derivatives in the far-field region

All vector potentials occurring in (6.74) and (6.75) are of the same general form. Their approximate form in the far-field region, i.e. as $|\underline{r}| \to \infty$, determines the approximate form of the scattered field in the far-field region. As a typical example we consider the vector potential associated with a volume-source distribution and write it as (cf.(6.70))

$$\underline{\Pi}_V(\underline{r}) = \iiint_{V_1} G(\underline{r}-\underline{r}')\underline{J}_V(\underline{r}')dV(\underline{r}') \quad \text{with } \underline{r} \in \mathbb{R}^3. \tag{6.B1}$$

The function $G(\underline{r}-\underline{r}')$ is given by (cf.(6.72))

$$G(\underline{r}-\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1}\exp(ik|\underline{r}-\underline{r}'|), \tag{6.B2}$$

where we have written

$$k \stackrel{\text{def}}{=} [(\sigma - i\omega\varepsilon)\, i\omega\mu]^{\frac{1}{2}}, \tag{6.B3}$$

with $\mathrm{Re}(k) > 0$ and $\mathrm{Im}(k) \geq 0$. First we construct the approximating expression for $G(\underline{r}-\underline{r}')$ as $|\underline{r}| \to \infty$. To this end we observe that from

$$|\underline{r}-\underline{r}'| = (\underline{r}\cdot\underline{r} - 2\underline{r}\cdot\underline{r}' + \underline{r}'\cdot\underline{r}')^{\frac{1}{2}}$$

$$= |\underline{r}|[1-2(\underline{r}\cdot\underline{r}')/|\underline{r}|^2 + |\underline{r}'|^2/|\underline{r}|^2]^{\frac{1}{2}} \tag{6.B4}$$

it follows that

$$|\underline{r}-\underline{r}'| = |\underline{r}| - \underline{\theta}\cdot\underline{r}' + \text{vanishing terms as } |\underline{r}| \to \infty, \tag{6.B5}$$

where

$$\underline{\theta} = \underline{r}/|\underline{r}|. \tag{6.B6}$$

Fig.6.8 illustrates (6.B5).With the aid of (6.B5) we approximate $G(\underline{r}-\underline{r}')$ by

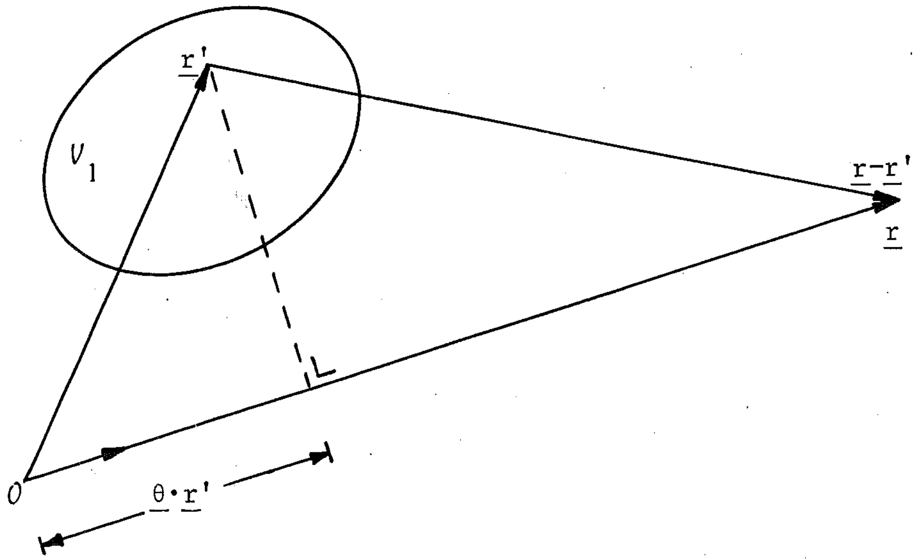

Fig 6.8.Position vectors $\underline{r}$ *and* $\underline{r}'$ *for a volume-source distribution; in the far-field region we have* $|\underline{r}-\underline{r}'| = |\underline{r}| - \underline{\theta}\cdot\underline{r}'$ *+ vanishing terms as* $|\underline{r}| \to \infty$.

$$G(\underline{r}-\underline{r}') \sim (4\pi|\underline{r}|)^{-1}\exp(ik|\underline{r}|)\exp(-ik\underline{\theta}\cdot\underline{r}') \quad \text{as } |\underline{r}| \to \infty. \tag{6.B7}$$

Substitution of (6.B7) in (6.B1) yields

$$\underline{\Pi}_V(\underline{r}) \sim \underline{\tilde{J}}_V(k\underline{\theta})(4\pi|\underline{r}|)^{-1}\exp(ik|\underline{r}|) \quad \text{as } |\underline{r}| \to \infty, \tag{6.B8}$$

where

$$\underline{\tilde{J}}_V(k\underline{\theta}) = \iiint_{V_1} \exp(-ik\underline{\theta}\cdot\underline{r}')\underline{J}_V(\underline{r}')dV(\underline{r}') \quad \text{with } \underline{\theta} \in \Omega. \tag{6.B9}$$

Note that (6.B9) is in accordance with (6.A1). Equation (6.B8) shows that the spherical-wave amplitude of $\underline{\Pi}_V(\underline{r})$ in the far-field region is the spatial Fourier transform $\underline{\tilde{J}}_V(\underline{k})$ of $\underline{J}_V(\underline{r})$, extended over the domain V_1 and taken at the value $\underline{k} = k\underline{\theta}$. Similar results hold for the vector potentials associated with surface-source distributions.

In the integral relations (6.74) and (6.75) pertaining to the electric- and the magnetic-field strengths also spatial derivatives of the vector potentials occur. To obtain their approximate representation in the far-field region we observe that

$$\underline{\nabla} G(\underline{r}-\underline{r}') = (ik - |\underline{r}-\underline{r}'|^{-1}) G(\underline{r}-\underline{r}') \underline{\nabla} |\underline{r}-\underline{r}'|, \qquad (6.B10)$$

where

$$\underline{\nabla} |\underline{r}-\underline{r}'| = (\underline{r}-\underline{r}')/|\underline{r}-\underline{r}'|. \qquad (6.B11)$$

However

$$(\underline{r}-\underline{r}')/|\underline{r}-\underline{r}'| = \underline{\theta} + \text{vanishing terms} \quad \text{as } |\underline{r}| \to \infty \qquad (6.B12)$$

and hence

$$\underline{\nabla} G(\underline{r}-\underline{r}') \sim ik\underline{\theta} G(\underline{r}-\underline{r}') \text{ as } |\underline{r}| \to \infty, \qquad (6.B13)$$

in which the expression (6.B7) for $G(\underline{r}-\underline{r}')$ is to be substituted. Using (6.B13) we arrive at

$$\underline{\nabla}[\underline{\nabla} \cdot \underline{\Pi}_V(\underline{r})] \sim ik\underline{\theta}[ik\underline{\theta} \cdot \underline{\Pi}_V(\underline{r})] \text{ as } |\underline{r}| \to \infty \qquad (6.B14)$$

and

$$\underline{\nabla} \times \underline{\Pi}_V(\underline{r}) \sim ik\underline{\theta} \times \underline{\Pi}_V(\underline{r}) \text{ as } |\underline{r}| \to \infty, \qquad (6.B15)$$

in which the expression (6.B8) for $\underline{\Pi}_V(\underline{r})$ is to be substituted. Again, similar results hold for the vector potentials associated with surface-source distributions.