Reciprocity, Discretization, and the Numerical Solution of Direct and Inverse Electromagnetic Radiation and Scattering Problems

ADRIANUS TEUNIS DE HOOP

Invited Paper

Except for the canonical problems in electromagnetics whose solution can be expressed in terms of analytic functions of a not too complicated nature, and for analytic approximation techniques (usually of an asymptotic nature) that can be applied to a wider variety of cases, wave propagation and scattering problems in electromagnetics have to be addressed with the aid of numerical techniques. Many of these methods can be envisaged as being discretized versions of appropriate "weak" formulations of the pertinent operator (differential or integral) equations. For the relevant problems as formulated in the time Laplace-transform domain it is shown that the Lorentz reciprocity theorem encompasses all known weak formulations, while its discretization leads to the discretized forms of the corresponding operator equations, in particular to their finite-element and integral-equation modeling schemes. Both direct (forward) and inverse problems are discussed.

I. Introduction

Except for the canonical problems in electromagnetics whose solution can be expressed in terms of analytic functions of a not too complicated nature (examples of which can be found in Bowman, Senior, and Uslenghi [1]) and for analytic approximation techniques (usually of an asymptotic nature), both in the long and the short wavelength regimes, that can be applied to a wider variety of cases [2], radiation and scattering problems in electromagnetics have to be addressed with the aid of numerical methods. Many of these methods can be envisaged as being discretized versions of appropriate "weak" formulations of the pertinent operator (differential or integral) equations. Specifically, the finite-element method, based on the method of weighted residuals [3], applied to the electromagnetic field equations, and the collocation method (method of point matching) applied to the source-type electromagnetic integral relations [4]–[6] can be grouped in this category. The present contribution is an attempt to systematize the different approaches and to bring, to a certain extent, consistency and coherence in the procedures.

To this end, the time Laplace-transform domain (s domain or complex frequency domain) electromagnetic Lorentz reciprocity theorem for time-invariant configurations is taken as point of departure. By taking the transform parameter s to be positive and real (as is done in the Cagniard method for calculating impulsive waves in stratified media [7], or complex in the right half $\mathrm{Re}(s) > 0$ of the complex s plane, the causality of the wave motion is ensured by requiring the time Laplace-transform domain wave-field quantities to be bounded functions of position in space, especially at infinitely large distances from the sources (of bounded extent) that generate the wave field. Also, arbitrarily inhomogeneous and anisotropic materials with arbitrary relaxation behavior can in this way be incorporated in the analysis in an easy manner. From the s-domain formulations the time-domain counterparts easily follow upon using some standard rules of the one-sided Laplace transformation, while the results for the limiting case of sinusoidally in time varying field quantities follow upon replacing s by $j\omega$, where j is the imaginary unit and ω the angular frequency of the oscillations, on the condition that imaginary values of s are approached via the right half of the complex s plane.

Taking one of the two states in the reciprocity theorem to be the electromagnetic state to be actually computed, and the other to be an appropriately chosen "auxiliary" or "computational" one, it is shown that the reciprocity theorem encompasses all known "weak" formulations of the

Manuscript received May 10, 1990; revised January 10, 1991. This work was supported in part by a Research Grant from the Stichting Fund for Science, Technology and Research (a companion organization to the Schlumberger Foundation in the U.S.), by a Research Grant from Schlumberger-Doll Research, Ridgefield, CT, and by a Research Grant from Etudes et Productions Schlumberger, Clamart, France.

The author is with the Laboratory of Electromagnetic Research, Faculty of Electrical Engineering, Delft University of Technology, 2600 GA Delft, The Netherlands.

IEEE Log Number 9103378.

0018-9219/91$01.00 © 1991 IEEE

electromagnetic-field differential equations and source-type integral relations. For this reason, the Lorentz reciprocity theorem is considered to best serve as the point of departure for the computational modeling of electromagnetic wave fields. (Note that this theorem is global rather than local in nature.)

Turning to numerics, first a geometrical discretization procedure is applied, with the tetrahedron (simplex in three-dimensional space) as the basic building block. The mesh size (supremum of the maximum diameters of the tetrahedra) of the discretization should be consistent with both the geometry of the configuration and the (inhomogeneous) distribution of matter in it, in the sense that it guarantees that the discretized configuration and the actual one differ relatively not more than a given fractional number according to some agreed-upon relative error criterion (for example, the normalized root-mean-square error of the difference in the material parameters). Subsequently, the wave-field quantities are expanded in terms of a base in an appropriately chosen linear function space, and a weighting procedure with an appropriately chosen "computational state" is carried out. This procedure leads to a system of linear algebraic equations in the expansion coefficients of the wave-field quantities. To solve the system of equations in the expansion coefficients, again, an error criterion is needed to define what the "best approximation" to their (iterative) solution is [8]. In this manner, several versions of the finite-element method can be understood as well as certain discretized versions of the integral equations describing scattering phenomena. Finally, it is remarked that both direct (forward) and inverse modeling are natural consequences of the reciprocity theorem in [9].

Through an analysis of the type presented here, it is, at least theoretically, feasible that the whole operation of arriving at numerical results of a predetermined accuracy is computer controlled.

II. The Electromagnetic Field in the Configuration

The configuration in which the electromagnetic field is present consists of a medium (or vacuum) that occupies three-dimensional space R^3. In the bounded subdomain $D^s \subset R^3$ the medium is, in general, inhomogeneous and anisotropic. The boundary surface ∂D^s of D^s is assumed to be piecewise smooth. In the unbounded domain $D_0 = \bar{D}^s$ that is the complement of the closure of D^s in R^3, the medium is homogeneous, isotropic, and lossless.

The position of observation in the configuration is specified by the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with origin O and the three mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ of unit length each. In the indicated order, the base vectors form a right-handed system. To cope notationally easily with the effects of anisotropy, the subscript notation for Cartesian vectors and tensors is used and the summation convention applies. The corresponding lowercase Latin subscripts are to be assigned the values {1, 2, 3}. This notation has also its advantages in view of the close resemblance between the resulting expressions and their corresponding statements in any of the high-level programming languages. Whenever appropriate, the position vector will be denoted by $\boldsymbol{x} = x_m \boldsymbol{i}_m$. The time coordinate is denoted by t. Partial differentiation is denoted by ∂; ∂_m denotes differentiation with respect to x_m, ∂_t is a reserved symbol for differentiation with respect to t.

For any causal space-time function $u = u(\boldsymbol{x}, t)$ the one-sided Laplace transform is introduced as

$$\hat{u}(\boldsymbol{x}, s) = \int_{t=0}^{\infty} \exp(-st) u(\boldsymbol{x}, t) dt \tag{1}$$

where the instant $t = 0$ marks the onset of the events. Obviously, for bounded $|u(\boldsymbol{x}, t)|$, $\hat{u}(\boldsymbol{x}, s)$ is an analytic function of the complex transform parameter s in the right half $\mathrm{Re}(s) > 0$ of the complex s plane. For ease of notation the circumflex over a symbol denoting its s-domain counterpart will be omitted in the remainder of the paper. The electromagnetic wave motion is started from a configuration at rest; then, under the one-sided Laplace transformation the operator ∂_t is replaced by an algebraic factor s.

In each subdomain of the configuration where the medium's electromagnetic properties vary continuously with position, the electromagnetic field quantities are continuously differentiable and satisfy the s-domain electromagnetic field equations

$$-\boldsymbol{\varepsilon}_{k,m,p} \partial_m H_p + \eta_{k,r} E_r = -J_k \tag{2}$$

$$\boldsymbol{\varepsilon}_{j,m,r} \partial_m E_r + \zeta_{j,p} H_p = -K_j \tag{3}$$

in which $\boldsymbol{\varepsilon}_{k,m,p}$ is the completely antisymmetric unit tensor of rank three (Levi–Civita tensor): $\boldsymbol{\varepsilon}_{k,m,p} = \{-1, +1\}$ if $\{k, m, p\}$ is an {odd, even} permutation of $\{1, 2, 3\}$ and $\boldsymbol{\varepsilon}_{k,m,p} = 0$ if not all subscripts are different, E_r is the electric field strength, H_p is the magnetic field strength, $\eta_{k,r}$ is the medium's transverse admittance per length ($\eta_{k,r} = \sigma_{k,r} + s\epsilon_{k,r}$, where $\sigma_{k,r}$ is the time Laplace transform of the medium's conductivity relaxation function, $\epsilon_{k,r}$ is the time Laplace transform of the medium's permittivity relaxation function; for an instantaneously reacting medium $\sigma_{k,r}$ and $\epsilon_{k,r}$ are the medium's s-independent conductivity and permittivity), $\zeta_{j,p}$ is the medium's longitudinal impedance per length ($\zeta_{j,p} = s\mu_{j,p}$, where $\mu_{j,p}$ is the time Laplace transform of the medium's permeability relaxation function; for an instantaneously reacting medium, $\mu_{j,p}$ is the medium's s-independent permeability), J_k is the volume source density of electric current, and K_j is the volume source density of magnetic current. Full anisotropy in the properties of the medium is taken into account. For media with relaxation, $\sigma_{k,r}$, $\epsilon_{k,r}$, and $\mu_{j,p}$ are s-dependent, subject to the condition of causality which entails analyticity in $\mathrm{Re}(s) > 0$. Across interfaces between different media, $\boldsymbol{\varepsilon}_{k,m,p} \nu_m H_p$ and $\boldsymbol{\varepsilon}_{j,m,r} \nu_m E_r$ are to be continuous, where ν_m is the unit vector along the normal to such an interface. In the configuration, objects can be present that are impenetrable to the electromagnetic field. On the boundary surface of such an object explicit boundary conditions hold; these are: either $\varepsilon_{j,m,r}\nu_m E_r \to 0$ (electrically impenetrable object) or $\varepsilon_{k,m,p}\nu_m H_p \to 0$ (magnetically impenetrable object).

In the domain D_0, which is denoted as the embedding, the medium is assumed to be homogeneous, isotropic and lossless, and we have $\eta_{k,r} = s\epsilon_0\delta_{k,r}$ and $\zeta_{j,p} = s\mu_0\delta_{j,p}$, in which ϵ_0 and μ_0 are position- and s-independent positive constants, and $\delta_{k,r}$ is the unit tensor of rank two (Kronecker tensor: $\delta_{k,r} = 1$ if $k = r$, $\delta_{k,r} = 0$ if $k \neq r$). The electromagnetic field quantities that are generated by known sources in such a medium are analytically known [10]. In deriving the relevant representations causality plays, again, an essential role.

III. The Contrast-Source or Scattering Formulation of the Electromagnetic Field Problem

The assumed simple electromagnetic properties of the embedding enable us to reformulate the electromagnetic wave problem as a contrast-source or scattering problem. To this end, it is assumed that in some bounded domain $D^i \subset D_0$ external sources with given volume distributions $\{J_k^i, K_j^i\}$ generate the electromagnetic field. The field $\{E_r^i, H_p^i\}$ that would be generated by these source distributions if also the domain D^s had the electromagnetic properties of the embedding D_0, is denoted as the incident field. Obviously, it satisfies the electromagnetic field equations

$$-\varepsilon_{k,m,p}\partial_m H_p^i + s\epsilon_0 E_k^i = \{-J_k^i, 0\} \quad \text{for } \boldsymbol{x} \in \{D^i, \bar{D}^i\} \tag{4}$$

$$\varepsilon_{j,m,r}\partial_m E_r^i + s\mu_0 H_j^i = \{-K_j^i, 0\} \quad \text{for } \boldsymbol{x} \in \{D^i, \bar{D}^i\} \tag{5}$$

where $\bar{D}^i$ is the complement of the closure of D^i in R^3. The total electromagnetic field $\{E_r, H_p\}$ in the configuration satisfies on account of (2) and (3) the electromagnetic field equations

$$-\varepsilon_{k,m,p}\partial_m H_p + \eta_{k,r}E_r = \{-J_k^i, 0\} \quad \text{for } \boldsymbol{x} \in \{D^i, \bar{D}^i\} \tag{6}$$

$$\varepsilon_{j,m,r}\partial_m E_r + \zeta_{j,p}H_p = \{-K_j^i, 0\} \quad \text{for } \boldsymbol{x} \in \{D^i, \bar{D}^i\}. \tag{7}$$

Upon introducing the scattered field $\{E_r^s, H_p^s\}$ as the difference between the total field and the incident field, i.e.,

$$\{E_r^s, H_p^s\} = \{E_r - E_r^i, H_p - H_p^i\}, \tag{8}$$

it follows from (4)–(5) and (6)–(7) that this wave field satisfies the electromagnetic field equations:

$$-\varepsilon_{k,m,p}\partial_m H_p^s + s\epsilon_0 E_k^s = \{-J_k^s, 0\} \quad \text{for } \boldsymbol{x} \in \{D^s, \bar{D}^s\} \tag{9}$$

$$\varepsilon_{j,m,r}\partial_m E_r^s + s\mu_0 H_j^s = \{-K_j^s, 0\} \quad \text{for } \boldsymbol{x} \in \{D^s, \bar{D}^s\} \tag{10}$$

where the contrast volume source densities $\{J_k^s, K_j^s\}$ of electric and magnetic current, respectively, are given by

$$J_k^s = (\eta_{k,r} - s\epsilon_0\delta_{k,r})E_r \tag{11}$$

$$K_j^s = (\zeta_{j,p} - s\mu_0\delta_{j,p})H_p. \tag{12}$$

Of course, the values of the contrast volume source densities are not known as long as the values of the total electromagnetic field in the contrasting domain D^s have not been determined.

IV. The Electromagnetic Reciprocity Relation

For our further analysis, the s-domain reciprocity relation that is associated with (2) and (3) will serve as the point of departure. A general wave-field reciprocity theorem interrelates, in a specific manner, the quantities that characterize two different physical states that could occur in one and the same domain in space-time. For time-invariant configurations the application of the one-sided Laplace transformation of (1) to the convolution-type reciprocity theorem leads to an equivalent s-domain result. For this to be applicable to the configuration under investigation, the media in the two states should be present in one and the same time-invariant subdomain D of the configuration. The two states will be distinguished by the superscripts A and B, respectively. The reciprocity relation will be arrived at by combining certain weighted forms, over the domain D, of the electromagnetic field equations pertaining to the two states.

First, the electromagnetic field equations (2) and (3) pertaining to the State A are multiplied through by the electric field strength and the magnetic field strength, respectively, pertaining to the State B, and integrated over the domain D. The result is

$$-\int_{\boldsymbol{x}\in D} E_k^B \varepsilon_{k,m,p}\partial_m H_p^A dV + \int_{\boldsymbol{x}\in D} E_k^B \eta_{k,r}^A E_r^A dV = -\int_{\boldsymbol{x}\in D} E_k^B J_k^A dV \tag{13}$$

$$\int_{\boldsymbol{x}\in D} H_j^B \varepsilon_{j,m,r}\partial_m E_r^A dV + \int_{\boldsymbol{x}\in D} H_j^B \zeta_{j,p}^A H_p^A dV = -\int_{\boldsymbol{x}\in D} H_j^B K_j^A dV. \tag{14}$$

Secondly, the electromagnetic field equations (2) and (3) pertaining to the State B are multiplied through by the electric field strength and the magnetic field strength, respectively, pertaining to the State A, and integrated over the domain D. The result is

$$-\int_{\boldsymbol{x}\in D} E_k^A \varepsilon_{k,m,p}\partial_m H_p^B dV + \int_{\boldsymbol{x}\in D} E_k^A \eta_{k,r}^B E_r^B dV = -\int_{\boldsymbol{x}\in D} E_k^A J_k^B dV \tag{15}$$

$$\int_{\boldsymbol{x}\in D} H_j^A \varepsilon_{j,m,r}\partial_m E_r^B dV + \int_{\boldsymbol{x}\in D} H_j^A \zeta_{j,p}^B H_p^B dV = -\int_{\boldsymbol{x}\in D} H_j^A K_j^B dV. \tag{16}$$

Equations (13)–(14) and (15)–(16) can be regarded as weighted forms of the electromagnetic field equations. Assuming that in D the electromagnetic properties of the medium vary continuously with position (under which condition the electromagnetic field quantities are continuously differentiable in D), taking the sum of (13) and (16), subtracting from the result the sum of (14) and (15), and applying Gauss' divergence theorem to the terms containing the spatial derivatives, it follows that

$$\begin{aligned}\varepsilon_{k,m,j}\int_{\boldsymbol{x}\in\partial D}&\nu_m(E_k^A H_j^B - E_k^B H_j^A)dA\\ &= \int_{\boldsymbol{x}\in D}[(\eta_{r,k}^B - \eta_{k,r}^A)E_r^A E_k^B\\ &\quad - (\zeta_{p,j}^B - \zeta_{j,p}^A)H_p^A H_j^B]dV\\ &\quad + \int_{\boldsymbol{x}\in D}(E_r^A J_r^B - E_k^B J_k^A + H_p^B K_p^A - H_j^A K_j^B)dV\end{aligned} \tag{17}$$

in which ∂D is the boundary surface of the domain D, which is assumed to be piecewise smooth and ν_m is the unit vector along its normal, pointing away from D. The validity of (17) is extended to a domain in which the electromagnetic medium properties are only piecewise continuously differentiable by adding the results of the (finite number of) subdomains to which (17) applies. In this procedure, the contributions from interfaces cancel in view of the pertaining boundary conditions of the continuity type, while the contributions from the boundary surfaces of impenetrable objects vanish in view of the pertaining boundary conditons of the explicit type. The thus obtained (17) is the global Lorentz reciprocity relation that will be used in the considerations that follow. Since, in it, spatial differentiations no longer occur, it can be regarded as the "weakest" form of the electromagnetic field equations (2) and (3).

Note that in the first integral on the right-hand side the differences in medium properties in the States A and B occur. This term vanishes if we take $\eta_{r,k}^B = \eta_{k,r}^A$ and $\zeta_{p,j}^B = \zeta_{j,p}^A$ at all points of D. If these conditions hold, the media in the States A and B are denoted as each others' adjoints. If the conditions hold for one and the same medium, such a medium is denoted as self-adjoint. In the second integral on the right-hand side the volume source distributions in the States A and B occur. This term vanishes if the domain D is source free. From the derivation it is clear that, due to the arbitrariness of the States A and B, subject of course to (2) and (3), (17) and (13)–(16) are equivalent as long as the pertaining boundary conditions are satisfied, both in State A and in State B.

In some of the applications, D will be the entire three-dimensional space. To address this situation, (17) is first applied to the domain interior to the sphere S_Δ of radius $\Delta > 0$ and with center at the origin of the chosen reference frame, after which the limit $\Delta \to \infty$ is taken. From some Δ onward, S_Δ will be entirely situated in the homogeneous, isotropic, lossless medium of the embedding. In view of this, the far-field representations for the electromagnetic field quantities can on S_Δ, for sufficiently large values of Δ, be used. From the latter, the contribution from S_Δ can be shown to vanish in the limit $\Delta \to \infty$ (cf. De Hoop [10]).

V. Discretization Procedure

The numerical handling of electromagnetic field and wave problems always implies that some discretized version of the wave problem is used to "approximate" the actual analytic one. In the following, the global reciprocity relation (17) will serve as the point of departure to define in what sense such an approximation holds. First, it is observed that each quantity $Q = Q(\boldsymbol{x}, s)$ occurring in the electromagnetic field or wave problem (which can be a scalar, a vector, or a tensor of arbitrary rank) and defined on some domain D has, after discretization, a discretrized counterpart $[Q] = [Q](\boldsymbol{x}, s)$ defined on the discretized version $[D]$ of D. Further, the actual machine computations are finite in number, and can therefore only be carried out for some bounded computational domain $D \subset R^3$. In the scheme to be presented, the inhomogeneous part D^s of the configuration has to be entirely incorporated in D, so we take D^s to be a proper subset of D. Note that the electromagnetic field is defined in the entire R^3, which implies that D also contains some part of D_0, where the medium is homogeneous, isotropic and lossless. Without loss of generality we can therefore take the discretized version of the domain of computation identical to the actual one, i.e., $[D] = D$.

A. Discretization of the Computational Domain

The domain of computation D is discretized by taking it to be the union of a finite number of tetrahedra (simplices in R^3) that all have vertices, edges and faces in common (see Naber [11]). The vertices of the tetrahedra will also be denoted as the nodes of the (geometrical) mesh and the supremum h of the maximum diameters of the tetrahedra will be denoted as the mesh size.

Except for the Green's functions to be introduced in Section IX, each quantity $Q = Q(\boldsymbol{x}, s)$ occurring in the electromagnetic field or wave problem will, in the interior of each tetrahedron, be approximated by the linear interpolation of its values at the vertices. Let $\{\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \boldsymbol{x}(3)\}$ denote the position vectors of the vertices of the tetrahedron *SMPLX* (simplex), then the corresponding linear interpolation is given by

$$[Q](\boldsymbol{x}, s) = \sum_{I^V=0}^{3} A^Q(I^V, s)\lambda(I^V; \boldsymbol{x}) \text{ for } \boldsymbol{x} \in \mathit{SMPLX} \tag{18}$$

in which

$$A^Q(I^V, s) = Q(\boldsymbol{x}(I^V), s) \tag{19}$$

is the value of Q at the vertex with ordinal number I^V, and $\{\lambda(I^V; \boldsymbol{x}); I^V = 0, 1, 2, 3\}$ are the barycentric coordinates of $\boldsymbol{x}$ in *SMPLX*. The latter have the property

$$\lambda(I^V, \boldsymbol{x}(J^V)) = \{1, 0\} \text{ if } \{I^V = J^V, I^V \neq J^V\} \tag{20}$$

and are expressed in terms of the vectorial areas of the faces of *SMPLX* through

$$\lambda(I^V; \boldsymbol{x}) = 1/4 - (1/3V)(x_m - b_m)A_m(I^V) \quad \text{with } I^V = 0, 1, 2, 3 \tag{21}$$

where $A_m(I^V)$ is the outwardly oriented vectorial area of the face opposite to the vertex with ordinal number I^V.

B. Discretization of the Medium Parameters

In the discretization of the medium parameters a distinction must be made between subdomains in which these parameters vary continuously with position and subdomains in which surfaces of discontinuity in these parameters occur. It is assumed that across such surfaces of discontinuity in medium parameters, the parameter values jump by finite amounts. Especially in applications where accurate values of the field quantities up to these surfaces are needed (such as, for example, in the modeling of borehole measurement situations in exploration geophysics, in the modeling of antenna configurations and in the analysis of the Electromagnetic Compatibility of microelectronic devices), special measures have to be taken to model the behavior of these quantities accurately. In principle, the medium properties can jump across any face of any tetrahedron of the discretized geometry. To accommodate this feature, all nodes of the geometrical mesh are considered as multiple nodes, where the multiplicity of each node is equal to the number of tetrahedra that meet at that node. The values of the constitutive parameters at the vertices follow either from user-supplied input expressions that are spatially sampled in the interior of each tetrahedron close to each of its vertices (as is the case in direct or forward scattering problems) or from computationally derived values (as is the case in inverse scattering problems). Out of the thus constructed local expansions of the medium parameters, their global expansions over the domain of computation are composed. If in the latter procedure, at a particular node no discontinuity turns up, the multiple node is replaced by a simple one, with an associated single value of the relevant constitutive parameter. For a nonscalar constitutive parameter (such as the ones that describe the electromagnetic properties of an anisotropic medium) the components with respect to the background Cartesian reference frame are used in the discretization procedure. The relevant global expansions for the medium properties are written as

$$[\eta_{k,r}](\boldsymbol{x},s) = \sum_{I^\eta=1}^{N^\eta} A^\eta(I^\eta,s)\Phi^\eta_{k,r}(I^\eta,\boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (22)$$

where $\{\Phi^\eta_{k,r}(I^\eta,\boldsymbol{x}); I^\eta = 1,\ldots,N^\eta\}$ is the sequence of global expansion functions for the medium's transverse admittance per length and $\{A^\eta(I^\eta,s); I^\eta = 1,\ldots,N^\eta\}$ is the sequence of its global expansion coefficients, and

$$[\zeta_{j,p}](\boldsymbol{x},s) = \sum_{I^\zeta=1}^{N^\zeta} A^\zeta(I^\zeta,s)\Phi^\zeta_{j,p}(I^\zeta,\boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (23)$$

where $\{\Phi^\zeta_{j,p}(I^\zeta\boldsymbol{x}); I^\zeta = 1,\ldots,N^\zeta\}$ is the sequence of global expansion functions for the medium's longitudinal impedance per length and $\{A^\zeta(I^\zeta,s); I^\zeta = 1,\ldots,N^\zeta\}$ is the sequence of its global expansion coefficients.

C. Discretization of the Volume Source Densities

For the discretization of the volume source densities the same procedure as for the discretization of the medium parameters is followed. In the case of direct or forward modeling, the values of the volume source densities at the vertices of the tetrahedra out of which the discretized geometry is composed, follow from user-supplied input expressions that are sampled in the interior of each tetrahedron close to each of its vertices. In the modeling of inverse source problems their values are computationally constructed. Out of the thus constructed local expansions of the volume source densities, their global expansions over the domain of computation are composed. If in this procedure at a particular node no discontinuity shows up, the multiple node is replaced by a simple one, with an associated single value of the relevant volume source density. Here, too, for the nonscalar volume source densities the components with respect to the background Cartesian reference frame are used in the discretization procedure. The relevant global expansions for the volume source densities are written as

$$[J_k](\boldsymbol{x},s) = \sum_{I^J=1}^{N^J} A^J(I^J,s)\Phi^J_k(I^J,\boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (24)$$

where $\{\Phi^J_k(I^J,\boldsymbol{x}); I^J = 1,\ldots,N^J\}$ is the sequence of global expansion functions for the volume source density of electric current and $\{A^J(I^J,s); I^J = 1,\ldots,N^J\}$ is the sequence of its global expansion coefficients, and

$$[K_j](\boldsymbol{x},s) = \sum_{I^K=1}^{N^K} A^K(I^K,s)\Phi^K_j(I^K,\boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (25)$$

where $\{\Phi^K_j(I^K,\boldsymbol{x}); I^K = 1,\ldots,N^K\}$ is the sequence of global expansion functions for the volume source density of magnetic current and $\{A^K(I^K,s); I^K = 1,\ldots,N^K\}$ is the sequence of its global expansion coefficients.

D. Discretization of the Electromagnetic Field Quantities

In the discretization of the electromagnetic field quantities the situation is more complicated. Here, some components are by necessity continuous across an interface of discontinuity in material properties, while other components show a finite jump across such a discontinuity surface. To preserve accuracy in the computational results, it is necessary, both in the modeling of direct or forward problems and in the modeling of inverse problems, to take computational measures that enforce the continuity conditions across an interface (in machine precision) and leave the noncontinuous components free to jump by finite amounts. For this purpose, local expansions of the type of (18) have been developed where a nonscalar quantity at a vertex is expressed in terms of those of its tensor components that are continuous across an interface of discontinuity in material properties. These components are, in general, not the components with respect to the background Cartesian reference frame [12]. For the electromagnetic field quantities their components tangential to an interface are continuous across the interface, while their normal components show a jump discontinuity. To guarantee the continuity of the tangential components of the electric and magnetic field strengths across each face of adjoining tetrahedra, we consider each node as a multiple node and construct at each vertex the electric and magnetic field strengths out of their three values along the three edges that meet at that vertex and use the relevant values in the relevant local expansions. (In the terminology of the finite-element method we have denoted such elements as "edge" elements [12]). Out of the thus constructed local expansions, the global expansions over the domain of computation are composed. If in this procedure simple nodes are met, a field strength is just expressed in terms of its components in the background Cartesian reference frame. The edge-element representation for the global expansion of the field strengths could also be used at simple nodes, but this leads to an unnecessarily larger number of expansion coefficients to be computed (without yielding an increased accuracy) since the number of tetrahedral edges that meet at a particular node is larger than three. The relevant global expansions for the field quantities are written as

$$[E_r](\boldsymbol{x}, s) = \sum_{I^E=1}^{N^E} A^E(I^E, s)\Phi_r^E(I^E, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (26)$$

where $\{\Phi_r^E(I^E, \boldsymbol{x}); I^E = 1, \ldots, N^E\}$ is the sequence of global expansion functions for the electric field strength and $\{A^E(I^E, s); I^E = 1, \ldots, N^E\}$ is the sequence of its global expansion coefficients, and

$$[H_p](\boldsymbol{x}, s) = \sum_{I^H=1}^{N^H} A^H(I^H, s)\Phi_p^H(I^H, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \quad (27)$$

where $\{\Phi_p^H(I^H, \boldsymbol{x}); I^H = 1, \ldots, N^H\}$ is the sequence of global expansion functions for the magnetic field strength and $\{A^H(I^H, s); I^H = 1, \ldots, N^H\}$ is the sequence of its global expansion coefficients.

VI. Source-Type Wave-Field Integral Representations For the Embedding

In an unbounded, homogeneous, isotropic, lossless medium with permittivity ϵ_0 and permeability μ_0 (like the one in the embedding D_0) the s-domain source-type wave-field representations are analytically known. They are given by (see also [10])

$$E_r = (s\epsilon_0)^{-1}(\partial_r\partial_k A_k - s^2\epsilon_0\mu_0 A_r) - \varepsilon_{r,m,j}\partial_m F_j \quad (28)$$

$$H_p = (s\mu_0)^{-1}(\partial_p\partial_j F_j - s^2\epsilon_0\mu_0 F_p) + \varepsilon_{p,m,k}\partial_m A_k \quad (29)$$

where

$$\{A_k, F_j\}(\boldsymbol{x}, s) = \int_{\boldsymbol{x}' \in D^T} G(\boldsymbol{x} - \boldsymbol{x}', s)\{J_k^T, K_j^T\}(\boldsymbol{x}', s)dV \quad (30)$$

are the electromagnetic vector potentials associated with the volume source distributions J_k^T of electric current and K_j^T of magnetic current, respectively,

$$G(\boldsymbol{x}) = \exp(-s|\boldsymbol{x}|/c_0)/4\pi|\boldsymbol{x}| \text{ for } \boldsymbol{x} \neq \boldsymbol{0} \quad (31)$$

is the infinite-medium Green's function of the modified Helmholtz equation

$$\partial_m\partial_m G - (s^2/c_0^2)G = -\delta(\boldsymbol{x}) \quad (32)$$

the quantity

$$c_0 = (\epsilon_0\mu_0)^{-1/2} \quad (33)$$

is the electromagnetic wave speed in the medium and D^T is the bounded spatial support of the volume source distributions. Representations of the type of (28)–(29) (or their discretized counterparts) play a vital role in the hybrid finite-element modeling to be discussed in Section VIII. In that kind of modeling, also the electromagnetic field equations (see also (4)–(8)):

$$-\varepsilon_{k,m,p}\partial_m H_p^s + \eta_{k,r}E_r^s = -\{(\eta_{k,r} - s\epsilon_0\delta_{k,r})E_r^i, 0\}, \quad \text{for } \boldsymbol{x} \in \{D^s, \bar{D}^s\} \quad (34)$$

$$\varepsilon_{j,m,r}\partial_m E_r^s + \zeta_{j,p}H_p^s = -\{(\zeta_{j,p} - s\mu_0\delta_{j,p})H_p^i, 0\} \quad \text{for } \boldsymbol{x} \in \{D^s, \bar{D}^s\} \quad (35)$$

satisfied by the scattered field, play a role.

VII. The Different Direct (Forward) and Inverse Problems

In the following we shall distinguish between the direct source, inverse source, direct scattering, and inverse scattering problems. The characteristics of these problems are summarized below.

A. Direct Source Problem

In the direct source problem, the constitutive coefficients (medium parameters) and the volume source distributions are known, and the values of the electromagnetic field quantities are to be found, at least in principle, in all space. The support of the volume source distributions that generate the electromagnetic wave field has the discretized version $[D^i]$. Depending on the computational modeling scheme to be employed, we have either $[D^i] \subset [D]$ or $[D^i] \subset [\bar{D}]$, where $[\bar{D}]$ is the complement of $[D] \cup \partial[D]$ in R^3.

B. Inverse Source Problem

In the inverse source problem, the constitutive coefficients of the medium are known, while in some domain $[D^\Omega] \subset [D]$ the values of the electromagnetic field quantities are assumed to be known from measurements, the electromagnetic radiation being assumed to be due to radiating sources with the bounded support $[D^T] \subset [D]$. The distribution of the volume source densities over $[D^T]$ is to be determined from the measured values of the electromagnetic field quantities in $[D^\Omega]$. Typically $[D^\Omega]$ and $[D^T]$ have no points in common.

C. Direct Scattering Problem

In the direct scattering problem, the constitutive coefficients of the medium and the volume source densities that generate the incident wave field are known, while the scattered wave field is to be found in all space. In view of (9) and (10) and the integral representations of the type of (28)–(29), the latter problem can be reduced to computing either the distribution of the quantities of the total electromagnetic field or the contrast source volume source densities (11) and (12) over the domain $[D^s]$, which is always chosen to be a proper subdomain of $[D]$.

D. Inverse Scattering Problem

In the inverse scattering problem the constitutive coefficients of the medium in the bounded domain whose discretized version is $[D^s]$, where $[D^s] \subset [D]$, are unknown, while this domain is embedded in a medium whose Green's functions (point-source electromagnetic wave fields) are analytically known. In particular, the latter applies to the chosen embedding D_0 of Section II. Further, in some domain $[D^\Omega] \subset [D]$ the values of the electromagnetic field quantities are assumed to be known from measurements. The constitutive coefficients in the domain $[D^s]$ are now to be reconstructed from the measured values of the electromagnetic field quantities in $[D^\Omega]$. Typically $[D^\Omega]$ and $[D^s]$ have no points in common.

In what follows we shall, to save on notation, collectively denote the medium parameters by M, the volume source densities by S and the wave-field quantities by F, and add these indications to the symbol Φ for the expansion functions and to the symbol A for the expansion coefficients, while the indication of the ranks of the different tensors involved is suppressed.

VIII. Hybrid Finite-Element Modeling

To construct the system of equations in the expansion coefficients of the pertaining unknown functions that result from the finite-element modeling of electromagnetic field problems, we apply either the weighted forms of the electromagnetic field equations (13)–(16) or the global Lorentz reciprocity relation (17) to the discretized computational domain $[D]$ of which D^s is taken to be a proper subdomain. For State A we take the representations of Section V applying to the volume source densities, the medium parameters and the (total or scattered, depending on whether $[D^i] \subset [D]$ or $[D^i] \subset [\bar{D}]$), electromagnetic field. State B is identified with successive, appropriately chosen, computational states. A feature of the finite-element modeling is that in all computations the medium parameters applying to the State B are taken to be zero, while the chosen wave fields in the State B, which are taken to be the successive elements of the expansion sequences of Section V, and the volume source densities in this state are (exactly) made to correspond to each other in accordance with (2) and (3). Further, on the boundary $[\partial D] = \partial[D]$ of the discretized computational domain $[D]$ "absorbing" boundary conditions are invoked that account for the fact that across it either the total electromagnetic field (in case the embedding is source free) or the scattered electromagnetic field (in case the sources that generate the incident field are situated in the embedding) radiates (causally) into the embedding.

In the so-called hybrid formulation of the finite-element method, the corresponding boundary values on $\partial[D]$ are taken to follow from the source-type integral representations (28) and (29), applied to the contrast-source electromagnetic field equations (9) and (10), discretizing these representations, taking the point of observation at the successive nodes of the boundary $\partial[D]$ and thus establishing a relationship between the values of $[E_r]$ and $[H_p]$ (or $[E_r^s]$ and $[H_p^s]$) on $\partial[D]$ and the values of $[E_k]$ and $[H_j]$ in $[D^s]$ that occur in the contrast volume source densities given by (11) and (12). This idea has been put forward by Lee, Pridmore, and Morrison [13]).

A. Direct Source Problem

At the interior nodes of the domain of computation $[D]$ we employ for the known constitutive parameters, the known volume source distributions and the unknown electromagnetic field quantities in State A the expansions that have been discussed in Section V. On the boundary $[\partial D] = \partial[D]$ of the domain of computation "absorbing" boundary conditions of the type indicated above are invoked. On the nodes coinciding with the boundary surfaces of impenetrable objects, the pertaining explicit boundary conditions are substituted. Depending on whether the source domain $[D^i]$, where the sources that generate the field are located, is part of $[D]$ or of $[\bar{D}]$, we proceed differently. The case $[D^i] \subset [D]$ is the simpler of the two. In this case the total wave field in $[D]$ is expanded according to Section V, i.e.,

$$[F^A] = [F] = \sum_{I^F=1}^{N^F} A^F(I^F, s)\Phi^F(I^F, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D]. \tag{36}$$

If $[D^i]$ is, however, located at a large distance from $[D^s]$, this procedure is impractical, since the domain of computation is then exceedingly large with an associated large demand for computer storage capacity and number of unknown expansion coefficients. In this case, we apply the scattering formulation of Section III in that we first compute the incident wave field from the integral representations (28) and (29) of Section VI, and next apply the finite-element formulation to the wave-field equations (34) and (35) for the scattered wave field. Correspondingly, the known right-hand sides are expanded according to Section V and for the scattered wave field in $[D]$ the expansion is written as

$$[F^A] = [F^s] = \sum_{I^{F;s}=1}^{N^{F;s}} A^{F;s}(I^{F;s}, s)\Phi^{F;s}(I^{F;s}, \boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in [D]. \tag{37}$$

Subsequently, for State B we put $[M_B] = 0$, take successively $[F^B] = \Phi(J^F, \boldsymbol{x})$, with $J^F = 1, \dots, N^F$, and match $[S^B]$ to these choices. Substituting everything in (13)–(16) or (17), a square system of linear, algebraic equations in $\{A^F(I^F, s); I^F = 1, \dots, N^F\}$ or $\{A^{F;s}(I^{F;s}, s); I^{F;s} = 1, \dots, N^{F;s}\}$ results, from which the relevant field expansion coefficients can be solved.

B. Inverse Source Problem

At the interior nodes of the domain of computation $[D]$ we employ for the known constitutive parameters, the unknown volume source distributions $[S^T]$ in their known support $[D^T] \subset [D]$, the known electromagnetic field quantities $[F^\Omega]$ in the domain of observation $[D^\Omega]$ and the unknown electromagnetic field quantities $[F]$ in $[D] \backslash [D^\Omega]$ in State A the expansions that have been discussed in Section V. On the boundary $[\partial D] = \partial[D]$ of the domain of computation "absorbing" boundary conditions of the type indicated above are invoked. On the nodes coinciding with the boundary surfaces of impenetrable objects, the pertaining explicit boundary conditions are substituted. In accordance with Section V the following expansions are used:

$$[S^A] = [S^T] = \sum_{I^S=1}^{N^S} A^S(I^S, s)\Phi^S(I^S, \boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in [D^T] \tag{38}$$

$$[F^A] = [F^\Omega] = \sum_{I^{F;\Omega}=1}^{N^{F;\Omega}} A^{F;\Omega}(I^{F;\Omega}, s)\Phi^{F;\Omega}(I^{F;\Omega}, \boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in [D^\Omega] \tag{39}$$

$$[F^A] = [F] = \sum_{I^F=1}^{N^F} A^F(I^F, s)\Phi^F(I^F, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D] \backslash [D^\Omega]. \tag{40}$$

Next, for State B we put $[M_B] = 0$, take successively $[F^B] = \Phi^{F;\Omega}(J^{F;\Omega}, \boldsymbol{x})$, with $J^{F;\Omega} = 1, \dots, N^{F;\Omega}$, with the correspondingly matched $[S^B]$ for $\boldsymbol{x} \in D^\Omega$ to produce known terms in (13)–(16) or (17), and $[F^B] = \Phi^F(J^F, \boldsymbol{x})$, with $J^F = 1, \dots, N^F$, with the correspondingly matched $[S^B]$ for $\boldsymbol{x} \in [D] \backslash D^\Omega$. Substituting everything in (13)–(16) or (17), a square system of linear, algebraic equations in $\{A^S(I^S, s); I^S = 1, \dots, N^S\}$ and $\{A^F(I^F, s); I^F = 1, \dots, N^F\}$ results if $N^{F;\Omega} = N^S + N^F$, from which system the relevant source expansion coefficients as well as the expansion coefficients for the wave field in $[D] \backslash [D^\Omega]$ can be solved, while an over determined system of linear, algebraic equations in $\{A^S(I^S, s); I^S = 1, \dots, N^S\}$ and $\{A^F(I^F, s); I^F = 1, \dots, N^F\}$ results if $N^{F;\Omega} > N^S + N^F$, from which system the "best" values of the source expansion coefficients and the expansion coefficients for the wave field in $[D] \backslash [D^\Omega]$ can be determined by minimizing the "error" in the satisfaction of the equality signs according to some error criterion.

C. Direct Scattering Problem

This problem is encompassed in the finite-element modeling of the direct source problem. To emphasize the scattering aspect of the problem, the computed electromagnetic field quantities in $[D^s] \subset [D]$ can be used to compute the contrast volume source densities of (11)–(12). From the latter, the scattered electromagnetic wave field everywhere in the configuration can be computed by using the integral representations (28)–(29).

D. Inverse Scattering Problem

This problem is most consistently dealt with by considering it as an inverse source problem, in the embedding, for the contrast volume source densities of (11)–(12), that have the support $[D^s]$ and generate the scattered electromagnetic wave field. After the sequences of expansion coefficients of these volume source densities have been computed, the resulting field computation problem can be formulated as a direct source problem whose solution leads to the values of the expansion coefficients of the scattered electromagnetic field in $[D^s]$, where the constitutive coefficients are still unknown. Subsequently, the reciprocity relation (17) is applied to the domain $[D^s]$ and (34)–(35). For the State A we take $[F^A] = [F^s]$, substitute for $[M^A]$ an expansion with N^M coefficients for the unknown constitutive coefficients of the type discussed in Section V, and identify $[S^A]$ with the discretized versions of the right-hand sides of (34)–(35) in which the incident field is known. For the State B we take successively a number of N^F local field states $[F^B] = \Phi^F(J^F, \boldsymbol{x})$, with $J^F = 1, \dots, N^F$, $[M^B] = 0$, and the correspondingly matched $[S^B]$. Then, a square system of linear, algebraic equations in the expansion coefficients of the medium parameters results if $N^F = N^M$, from which system the expansion coefficients of the medium parameters can be solved, while an over determined system of linear, algebraic equations in the expansion coefficients of the medium parameters results if $N^F > N^M$, from which the "best" values of the expansion coefficients of the medium parameters can be determined by minimizing the "error" in the satisfaction of the equality signs according to some error criterion.

IX. Integral-Equation Modeling

To construct the system of equations in the expansion coefficients of the pertaining unknown functions that result from the integral-equation modeling of electromagnetic field problems, we apply in a particular number of steps the global Lorentz reciprocity relation (17) to the discretized computational domain $[D]$ of which D^s is taken to be a proper subdomain. The integral-equation formulation is invariably based on the contrast-source description discussed in Section III and on source-type wave-field integral representations of the type given in Section VI for the case of a homogeneous, isotropic, lossless medium. In fact, the formulation can be extended to any case where the electromagnetic field Green's functions (point-source solutions to the electromagnetic field equations), or their discretized counterparts, can be constructed analytically.

The first step in the integral-equation formulation consists of constructing the radiated wave fields $[F^G] = G(\boldsymbol{x}, s; I^G)$ with $\{G(\boldsymbol{x}, s; I^G); \boldsymbol{x} \in R^3, I^G = 1, \ldots, N^G\}$ that correspond to the localized source distributions $[S^G] = \Phi^S(I^G, \boldsymbol{x}')$ for $\boldsymbol{x}' \in [D^G(I^G)]$, where $[D^G(I^G)]$ is some localized source domain of the geometrical discretization as discussed in Section V. It is assumed that these radiated wave fields are analytically evaluated from the expressions at the right-hand sides of (28)–(29). Next, the contrast volume-source densities $[S^s]$ are written as the expansions

$$[S^s](\boldsymbol{x}, s) = \sum_{I^{S;s}=1}^{N^{S;s}} A^{S;s}(I^{S;s}, s)\Phi^{S;s}(I^{S;s}, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D^s] \tag{41}$$

with the unknown coefficients $\{A^{S;s}(I^{S;s}, s); I^{S;s} = 1, \ldots, N^{S;s}\}$. The corresponding expansion for the scattered field is given by

$$[F^s](\boldsymbol{x}, s) = \sum_{I^{S;s}=1}^{N^{S;s}} A^{S;s}(I^{S;s}, s)G(\boldsymbol{x}, s; I^{S;s}) \text{ for } \boldsymbol{x} \in R^3. \tag{42}$$

A similar expansion holds for the incident field. Let the volume source densities that generate the incident field be expanded as

$$[S^i](\boldsymbol{x}, s) = \sum_{I^{S;i}=1}^{N^{S;i}} A^{S;i}(I^{S;i}, s)\Phi^{S;i}(I^{S;i}, \boldsymbol{x}) \text{ for } \boldsymbol{x} \in [D^i] \tag{43}$$

with the known coefficients $\{A^{S;i}(I^{S;i}, s); I^{S;i} = 1, \ldots, N^{S;i}\}$. Then, the corresponding expansion for the incident wave field is given by

$$[F^i](\boldsymbol{x}, s) = \sum_{I^{S;i}=1}^{N^{S;i}} A^{S;i}(I^{S;i}, s)G(\boldsymbol{x}, s; I^{S;i}) \text{ for } \boldsymbol{x} \in R^3. \tag{44}$$

The different problems enumerated in Section VII will now be discussed separately for the case of penetrable scatterers. The case of impenetrable scatterers runs along similar lines, be it that the contrast source representations of the volume type that follow from Section VI have to be replaced by contrast source representations of the surface type, in which the contrast sources are located at the boundary surfaces of the impenetrable objects.

A. Direct Source Problem

In the direct source problem the sources in $[D^i]$ radiate in the known embedding, and the incident field as given by (44) is already the total field, a scattered field being absent.

B. Inverse Source Problem

In the inverse source problem the unknown volume source distributions $[S^T]$ are expanded as

$$[S^T](\boldsymbol{x}, s) = \sum_{I^{S;T}=1}^{N^{S;T}} A^{S;T}(I^{S;T}, s)\Phi^{S;T}(I^{S;T}, \boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in [D^T] \tag{45}$$

in which $\Phi^{S;T}(I^{S;T}, \boldsymbol{x})$ is a local volume source expansion function of the discretized source domain $[D^T]$ and the coefficients $\{A^{S;T}(I^{S;T}, s); I^{S;T} = 1, \ldots, N^{S;T}\}$ are to be determined. The corresponding radiated wave field, which is measured in $[D^\Omega]$, is then given by

$$[F^T](\boldsymbol{x}, s) = \sum_{I^{S;T}=1}^{N^{S;T}} A^{S;T}(I^{S;T}, s)G(\boldsymbol{x}, s, I^{S;T}) \quad \text{for } \boldsymbol{x} \in [R^3] \tag{46}$$

where $\{G(\boldsymbol{x}, s; I^{S;T}); \boldsymbol{x} \in R^3, I^{S;T} = 1, \ldots, N^{S;T}\}$ are the radiated wave fields that correspond to the localized source distributions $\Phi^{S;T}(I^{S;T}, \boldsymbol{x})$ with supports $[D^T(I^{S;T})] \subset [D]$. Next, the reciprocity relation (17) is applied to the domain $[D^T] \cup [D^\Omega]$, while the medium parameters in the States A and B are both chosen to be the ones of the embedding, i.e., $[M^A] = [M^B] = [M_0]$. Further, we take $[S^A] = [S^T]$ and $[F^A] = [F^T]$. For State B we successively take $[F^B] = G(\boldsymbol{x}, s; J^{S;\Omega})$, where $\{G(\boldsymbol{x}, s; J^{S;\Omega}); \boldsymbol{x} \in R^3, J^{S;\Omega} = 1, \ldots, N^{S;\Omega}\}$ are the radiated wave fields that correspond to the localized source distributions $[S^\Omega] = \Phi^{S;\Omega}(I^{S;\Omega}, \boldsymbol{x})$ with supports $[D^\Omega(I^{S;\Omega})] \subset [D^\Omega]$. Substituting everything in (17), a square system of linear, algebraic equations in $\{A^{S;T}(I^{S;T}, s); I^{S;T} = 1, \ldots, N^{S;T}\}$ results if $N^{S;\Omega} = N^{S;T}$, from which system the relevant source expansion coefficients can be solved, while an over determined system of linear, algebraic equations in $\{A^{S;T}(I^{S;T}, s); I^{S;T} = 1, \ldots, N^{S;T}\}$ results if $N^{S;\Omega} > N^{S;T}$, from which system the "best" values of the source expansion coefficients can be determined by minimizing the "error" in the satisfaction of the equality signs according to some error criterion.

C. Direct Scattering Problem

In the direct scattering problem the reciprocity relation (17) is applied to the domain $[D^s]$. For State A we take the wave field to be the scattered field as it follows from the contrast-source equations (9)–(10), i.e., $[F^A] = [F^s]$, $[M^A] = [M_0]$, and $[S^A] = [S^s]$. For State B we take the volume source distributions to be successively one of the local discretized source distributions $[S^B] = [S^G] = \Phi^{S;s}(J^{S;s}, \boldsymbol{x})$ for $J^{S;s} = 1, \ldots, N^{S;s}$, whose corresponding wave fields in the medium $[M^B] = [M_0]$ are $[F^B] = G(\boldsymbol{x}, s, J^{S,s})$. Substituting everything in (17), using for $[S^s]$ and $[F^s]$ the expansions given by (41) and (42), respectively, and substituting in the right-hand sides of (11)–(12) the expansions applying to the known constitutive coefficients, the known incident wave field and the unknown scattered wave field, a square system of linear algebraic equations results from which the expansion coefficients $\{A^{S;s}(I^{S;s}, s); I^{S;s} = 1, \ldots, N^{S;s}\}$ of the contrast source distributions can be solved. Substituting these values in (42), the scattered wave field follows anywhere in space.

D. Inverse Scattering Problem

This problem is most consistently dealt with by considering it as an inverse source problem for the contrast volume source densities of (11)–(12), that have the support $[D^s]$ and generate the scattered electromagnetic field. After the sequences of expansion coefficients of these volume source densities have been computed, the scattered wave field follows from (42) anywhere in space. Next, the reciprocity relation (17) is applied to the domain $[D^s]$ with $[F^A] = [F^s]$, $[S^A] = [S^s]$, $[M^A] = [M_0]$, while for the unknown contrasts in the constitutive coefficients in the right-hand sides of (11)–(12) an expansion, with N^χ coefficients, of the type discussed in Section V are taken. For the State B we successively take a number N^G of discretized Green's states as introduced in the beginning of the present section: $[F^B] = [F^G]$, $[S^B] = [S^G]$, $[M^B] = [M_0]$. Subsequently applying the reciprocity relation (17) to the domain $[D^s]$, a system of linear, algebraic equations in the expansion coefficients of the contrasts in the constitutive coefficients results. For $N^G = N^\chi$ this is a square system from which the expansion coefficients for the contrasts in the constitutive coefficients can be solved; for $N^G > N^\chi$ this system is over determined and the "best" values of the expansion coefficients for the contrasts in the constitutive coefficients can be determined by minimizing the "error" in the satisfaction of the equality signs according to some error criterion. From the obtained values, the values of the expansion coefficients of the constitutive coefficients themselves follow directly.

X. Conclusion

It has been shown that the Lorentz reciprocity relation can be used as the point of departure for the numerical modeling of electromagnetic direct (or forward) and inverse wave propagation and scattering problems. In fact, all known numerical procedures (finite-element method, integral-equation method) are shown to be consequences of it.

References

[1] J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi, *Electromagnetic and acoustic scattering by simple shapes.* Amsterdam, The Netherlands: North-Holland, 1969.

[2] L. B. Felsen and N. Marcuvitz, *Radiation and scattering of waves.* Englewood Cliffs, NJ: Prentice-Hall, 1973.

[3] O. C. Zienkiewicz and R. L. Taylor, *The finite element method.* London: McGraw-Hill, 4th ed., 1989.

[4] A. J. Poggio and E. K. Miller, "Integral-equation solutions of three-dimensional scattering problems," in *Computational Techniques for Electromagnetics*, R. Mittra, Ed. Oxford, U.K.: Pergamon, 1973, ch. 4.

[5] R. F. Harrington, *Field Computation by Moment Methods.* New York: Macmillan, 1968.

[6] A. T. de Hoop, "General considerations on the integral-equation formulation of diffraction problems," in *Modern Topics in Electromagnetics and Antennas.* Stevenage: Peter Peregrinus, 1977, ch. 6.

[7] J. H. M. T. van der Hijden, *Propagation of transient elastic waves in stratified anisotropic media.* Amsterdam, The Netherlands: North-Holland, 1987.

[8] P. M. van den Berg, "Iterative schemes based on the minimization of the error in field problems," *Electromagnetics*, vol. 5 , pp. 237–262, 1985.

[9] A. T. de Hoop, "Time-domain reciprocity theorems for electromagnetic fields in dispersive media," *Radio Sci.*, vol. 22, no. 7, pp. 1171–1178, Dec. 1987.

[10] A. T. de Hoop, "A time-domain energy theorem for scattering of plane electromagnetic waves," *Radio Sci.*, vol. 19, no. 1, pp. 1179–1184, Sept.-Oct. 1984.

[11] L. Naber, *Topological methods in Euclidean space.* Cambridge, U.K.: Cambridge University Press, 1980.

[12] G. Mur and A. T. de Hoop, "A finite-element method for computing three-dimensional electromagnetic fields in inhomogeneous media," *IEEE Trans. Magnetics*, vol. MAG-21, no. 6, pp. 2188–2191, Nov. 1985.

[13] K. H. Lee, D. F. Pridmore, and H. F. Morrison, "A hybrid three-dimensional electromagnetic modeling scheme," *Geophysics*, vol. 46, no. 3, pp. 796–805, May 1981.

Adrianus Teunis de Hoop was born in Rotterdam, The Netherlands, in 1927. He attended the Delft University of Technology, where he received the M.Sc. degree in 1950 and the Ph.D. degree in 1958. He was granted an Honorary Doctor's Degree in the Applied Sciences from the State University at Ghent, Belgium in 1982.

From 1956 to 1957, he was a research assistant at the Institute of Geophysics, University of California at Los Angeles. He was an Associate Professor from 1957 to 1960 and became Professor in 1960 of Electromagnetic Theory and Applied Mathematics at the Delft University of Technology in Delft, The Netherlands. He participates on a regular basis in the Visiting Scientist Program of Schlumberger-Doll Research, Ridgefield, CT (1982) and Schlumberger Cambridge Research, Cambridge, England (1991). His major research area consists of fundamental aspects of acoustic, electromagnetic, and elastodynamic wave propagation and scattering.

Dr. de Hoop has received awards from the "Stichting Fund for Science, Technology and Research" (a companion organization to the Schlumberger Foundation in the U.S. from 1986 to 1990.) He was appointed a member of the Royal Netherlands Academy of Arts and Sciences and was presented with the "Speurwerkprijs" (Gold Research Medal) of the Royal Institution of Engineers in The Netherlands (1989). He is a member of The Electromagnetics Academy, the Acoustical Society of America, and the Society of Exploration Geophysics.