A concise introduction to the mathematics of economic analysis

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Abstract

A concise introduction to the major (deterministic) mathematical methods of economic analysis is presented, arranged as

- aims and scopes
- mathematical concepts
- mathematical tools (differential calculus)
- mathematical tools (integral calculus)
- mathematical tools (miscellaneous)
- special functions
- optimization techniques
- examples and applications

⟹ 01 ECONOMIC ANALYSIS - AIMS AND SCOPE ⟸

01.01 ECONOMIC ANALYSIS - AIMS AND SCOPE

ECONOMIC ANALYSIS aims at developing tools for **OPTIMIZING** (i.e. **MAXIMIZING** or **MINIMIZING**) economic or business processes via **MATHEMATICAL MODELS** that express the **FUNCTIONAL DEPENDENCE** of the **TARGETS** on their **ENDOGENOUS** factors ('variables') **EXOGENOUS** ('parameters'). The **OPTIMIZATION CRITERION** usually takes the form of a (sometimes large) number of **STRUCTURAL EQUATIONS** that interrelate the **OPTIMUM VALUES** of the variables for a given set of parameters.

01.02 OPTIMIZATION PROCEDURE

The **OPTIMIZATION PROCEDURE** takes place via an analysis of the **LOCAL DIFFERENTIAL BEHAVIOR** of the **TARGET FUNCTION**. This behavior is produced with the **TAYLOR EXPANSION** of the target function. This expansion is constructed with the aid of **DIFFERENTIATION** and **INTEGRATION** operations on the target function.

01.03 MODEL TARGET FUNCTIONS

Simple **MODEL TARGET FUNCTIONS** are combinations of **ALGEBRAIC FUNCTIONS** (sums, differences, products, quotients, powers) and **SIMPLE TRANSCENDENTAL FUNCTIONS** (exponentials, logarithms) of the parameters and variables.

01.04 STRUCTURAL EQUATIONS

For a large number of variables the bookkeeping of the **STRUCTURAL EQUATIONS** takes place via **MATRIX or ARRAY ALGEBRA**, for which efficient computer software exists (e.g. Matlab^{TM}).

$\Longrightarrow$ 02 MATHEMATICAL CONCEPTS $\Longleftarrow$

02.01 ABSOLUTE VALUE

For any $x \in \mathbb{R}$, the *'absolute value'* $|x|$ of x' is defined as

- $|x| = \{-x, 0, x\}$ for $\{x < 0, x = 0, x > 0\}$

Property:

- $|x + y| \leq |x| + |y|$

02.02 INEQUALITIES

For any $(a, b) \in \mathbb{R}$, either $a < b$ or $a = b$ or $a > b$ holds. (The real numbers form an *ordered sequence.*)

- $a < b \Longrightarrow a - b < 0 \Longrightarrow -b < -a$
- $a = b \Longrightarrow a - b = 0 \Longrightarrow -b = -a$
- $a > b \Longrightarrow a - b > 0 \Longrightarrow -b > -a$

02.03 POWERS

Powers ($x \neq 0$):

- $x^0 = 1$
- $x^n = x \cdot x^{n-1}$ for $n = 1, 2, 3, \ldots$
- $\left(\frac{x^m}{y^n}\right)^p = \left(x^m \cdot y^{-n}\right)^p = x^{p \cdot m} \cdot y^{-p \cdot n}$

02.04 FACTORIAL

Factorial $\{n!; n = 1, 2, 3, \dots\}$:

- $1! = 1$
- $n! = n \cdot (n-1)!$ for $n = 2, 3, \dots$
- $0! = 1$ by definition

02.05 SEQUENCES, SERIES

A sequence is an ordered set of numbers

$$\{x_1, \dots, x_N; N \geq 1\}$$

or

$$\{x_0, \dots, x_N; N \geq 0\}$$

Their (partial) sums are the series

$$S_1^n = \sum_{k=1}^{n} x_k \text{ for } 1 \leq n \leq N$$

or

$$S_0^n = \sum_{k=0}^{n} x_k \text{ for } 0 \leq n \leq N$$

Property:

- $$|S_1^n| \leq \sum_{k=1}^{n} |x_k| \text{ for } 1 \leq n \leq N$$
  $$\leq n \cdot \max_{\{1 \leq k \leq n\}}(|x_k|)$$

Property:

- $$|S_0^n| \leq \sum_{k=0}^{n} |x_k| \text{ for } 0 \leq n \leq N$$
  $$\leq (n+1) \cdot \max_{\{0 \leq k \leq n\}}(|x_k|)$$

02.06 ARITHMETIC PROGRESSION (ARITHMETIC SEQUENCE)

Arithmetic progression (arithmetic sequence) $\{a_n; n = 0, \ldots, N; d \in \mathbb{R}\}$:

$$\begin{aligned} a_n &= a_{n-1} + d \text{ for } n = 1, \ldots, N \\ a_0 &= \text{initial term} \\ d &= \text{common difference of successive members} \\ &\Longrightarrow \\ \bullet\ a_n &= a_0 + n \cdot d \text{ for } n = 0, 1, 2, \ldots \end{aligned}$$

02.07 ARITHMETIC SERIES (SUM)

Arithmetic series (sum) $\{a_n; n = 0, \ldots, N; d \in \mathbb{R}\}$:

$$\begin{aligned} S_N &= \sum_{n=0}^{N} a_n \\ &= \sum_{n=0}^{N} (a_0 + n \cdot d) \\ &= \sum_{n=0}^{N} (a_N - n \cdot d) \text{ (counting backwards)} \\ &\Longrightarrow \\ 2S_N &= (N+1) \cdot (a_0 + a_N) \\ &\Longrightarrow \\ \bullet\ S_N &= \frac{1}{2} \cdot (N+1) \cdot (a_0 + a_N) \end{aligned}$$

02.08 GEOMETRIC PROGRESSION (GEOMETRIC SEQUENCE)

Geometric progression (geometric sequence) $\{a_n; n = 0, \ldots, N; r \neq 0\}$:

$$\begin{aligned} a_n &= a_{n-1} \cdot r \text{ for } n = 1, \ldots, N \\ a_0 &= \text{scale factor} \\ r &= \text{common ratio of successive members} \\ &\Longrightarrow \\ \bullet\ a_n &= a_0 \cdot r^n \text{ for } n = 1, 2, 3, \ldots \end{aligned}$$

02.09 GEOMETRIC SERIES (SUM)

Geometric series (sum) $\{a_n; n = 0, \ldots, N; d \in \mathbb{R}\}$:

$$
\begin{aligned}
S_N &= \sum_{n=0}^{N} a_n \\
&= \sum_{n=0}^{N} a_0 \cdot r^n = a_0 \cdot \sum_{n=0}^{N} r^n = a_0 + a_0 \cdot \sum_{n=1}^{N} r^n \\
&\Longrightarrow \\
r \cdot S_N &= a_0 \cdot \sum_{n=0}^{N} r^{n+1} = a_0 \cdot \sum_{n=1}^{N} r^n + a_0 \cdot r^{N+1} \\
&\Longrightarrow \\
(1-r)S_N &= a_0 \cdot (1 - r^{N+1}) \\
&\Longrightarrow \\
\bullet\ S_N &= a_0 \cdot \frac{1 - r^{N+1}}{1 - r} \text{ for } r \neq 1
\end{aligned}
$$

Exercise: Show that

$$\frac{a^{N+1} - b^{N+1}}{a - b} = \sum_{n=0}^{N} a^{N-n} \cdot b^n \text{ for } N = 0, 1, 2, \ldots$$

(*Hint:* Use the sum formula for the geometric series with $a_0 = 0$ and $r = b/a$.)

Exercise: Show that

$$\frac{a^{N+1} + b^{N+1}}{a + b} = \sum_{n=0}^{N} (-1)^n \cdot a^{N-n} \cdot b^n \text{ for } N = 0, 1, 2, \ldots$$

(*Hint:* Use the sum formula for the geometric series with $a_0 = 0$ and $r = -b/a$.)

02.10 AVERAGES

Given is the sequence: • $\{a_1, \dots, a_N\}$
Arithmetic average (Arithmetic mean):

$$m_A = \frac{1}{N}\sum_{n=1}^{N} a_n$$

Geometric average (Geometric mean):

$$m_G = \left(\prod_{n=1}^{N} a_n\right)^{1/N}$$

Harmonic average (Harmonic mean):

$$\frac{1}{m_H} = \frac{1}{N}\sum_{n=1}^{N} \frac{1}{a_n}$$

Property: It can be shown (by induction) that

$$m_A \geq m_G \geq m_H$$

02.11 FUNCTIONS

$$\bullet\ y = f(x,p) : \{x \in \mathcal{D}_{\text{var}}, p \in \mathcal{D}_{\text{par}}\} \xmapsto{f(x,p)} \{y \in \mathcal{R}_{\text{fun}}\}$$

$\mathcal{D}_{\text{var}}$: domain of variables $\{x\}$

$\mathcal{D}_{\text{par}}$: domain of parameters $\{p\}$

$\mathcal{R}_{\text{fun}}$: range of function (operator) f

NOTE: In optimization theory, usually only the variable to be optimized is made explicit in the list of function arguments

02.12 VENN DIAGRAM

$$\boxed{\mathcal{D}_{\text{var}}} \cup \boxed{\mathcal{D}_{\text{par}}} \xmapsto{f} \boxed{\mathcal{R}_{\text{fun}}}$$

02.13 GRAPH

• $y = f(x,p)$: $x \longrightarrow$ horizontal axis, $y \longrightarrow$ vertical axis, $p \longrightarrow$ in figure text

02.14 LANDAU ORDER SYMBOLS O AND o

The Landau Order symbol $\boldsymbol{O}$

If two functions $f(x)$ and $g(x)$ are defined on the same domain $\mathcal{D}_{\mathrm{var}}$, the *'Landau Order symbol'* O expresses that

- $f(x) = O[g(x)]$ for $x \in \mathcal{D}_{\mathrm{var}}$

if for some $A > 0$ the relation

$$|f(x)| < A \cdot |g(x)| \text{ for all } x \in \mathcal{D}_{\mathrm{var}}$$

holds.

The Landau order symbol $\boldsymbol{o}$

If two functions $f(x)$ and $g(x)$ are defined on the same domain $\mathcal{D}_{\mathrm{var}}$ and $x_0 \in \mathcal{D}_{\mathrm{var}}$ is an interior point of $\mathcal{D}_{\mathrm{var}}$, the *'Landau order symbol'* o expresses that

- $f(x) = o[g(x)]$ as $x \to x_0$

if for some $m(|x - a|) > 0$ the relation

$$|f(x)| < m(|x - a|) \cdot |g(x)| \text{ as } x \to a$$

holds, where

$$m(|x - a|) \to 0 \text{ as } x \to a$$

02.15 LINEAR FUNCTION

The function

- $f(x) = a \cdot x + b$ with $a \neq 0$

is a *linear function* of x, defined on $\mathbb{R}$. Its (single) *zero* x_1 follows from

- $f(x_1) = 0$

as

- $x_1 = -\dfrac{b}{a}$

The *graph* of $f(x)$ is a *straight line*.

02.16 QUADRATIC FUNCTION

The function

$$\bullet\ f(x) = a \cdot x^2 + b \cdot x + c \text{ with } a \neq 0$$

is a *quadratic function* of x, defined on $\mathbb{R}$. Its two *zeros* x_1 and x_2 follow from

$$f(x) = a \cdot \left\{ \left(x + \frac{b}{2 \cdot a} \right)^2 - \left[\left(\frac{b}{2 \cdot a} \right)^2 - \frac{c}{a} \right] \right\}$$

and

$$\bullet\ f(x_1) = 0 \ ; \bullet\ f(x_2) = 0$$

as

$$\bullet\ x_1 = -\frac{b}{2 \cdot a} - \left(\frac{b^2}{4 \cdot a^2} - \frac{c}{a} \right)^{1/2} \text{ if } \frac{b^2}{4 \cdot a^2} - \frac{c}{a} \geq 0$$
$$\bullet\ x_2 = -\frac{b}{2 \cdot a} + \left(\frac{b^2}{4 \cdot a^2} - \frac{c}{a} \right)^{1/2} \text{ if } \frac{b^2}{4 \cdot a^2} - \frac{c}{a} \geq 0$$

If $b^2/4 \cdot a^2 - c/a < 0$, $f(x)$ has no (real-valued) zeros.
If $b^2/4 \cdot a^2 - c/a = 0$, the two zeros of$f(x)$ coincide.
The *stationary point* x_0 of $f(x)$ follows from

$$\bullet\ \partial_x f(x)\big|_{x=x_0} = 0$$

with

$$\partial_x f(x) = 2 \cdot a \cdot x + b$$

as

$$\bullet\ x_0 = -\frac{b}{2 \cdot a}$$

The *graph* of $f(x)$ is a *parabola*.

02.17 CAUCHY-SCHWARZ INEQUALITY

- The quadratic function of λ

  - $$\sum_{n=1}^{N}(\lambda \cdot a_n + b_n)^2 = \lambda^2 \cdot \left(\sum_{N=1}^{N} a_n^2\right) + 2 \cdot \lambda \cdot \sum_{N=1}^{N}(a_n \cdot b_n) + \left(\sum_{N=1}^{N} b_n^2\right)$$

is *non-negative* for any value of λ. This leads to (see Section 2.16):

  - $$\left(\sum_{N=1}^{N}(a_n \cdot b_n)\right)^2 \leq \left(\sum_{N=1}^{N} a_n^2\right) \cdot \left(\sum_{N=1}^{N} b_n^2\right)$$ (Cauchy-Schwarz inequality)

$\Longrightarrow$ 03 MATHEMATICAL TOOLS (DIFFERENTIAL CALCULUS) $\Longleftarrow$

03.01 DERIVATIVE OF A LINEAR FUNCTION ON A BOUNDED INTERVAL $\mathcal{X} : \{x \in \mathbb{R}; x_0 \le x \le x_1\}$

$$
\begin{aligned}
f(x) &= \frac{x_1 - x}{x_1 - x_0} \cdot f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot f(x_1) \text{ for } x \in \mathcal{X} \ (x_0 \le x \le x_1) \\
&\Longrightarrow \\
f(x_1) - f(x_0) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot (x_1 - x_0) \\
\mathrm{d}f &= f(x_1) - f(x_0) = \text{ differential of } f \\
\mathrm{d}x &= x_1 - x_0 = \text{ differential of } x \\
\frac{f(x_1) - f(x_0)}{x_1 - x_0} &= \partial_x f(x) = \text{ (first-order) derivative of } f \text{ for } x \in \mathcal{X} \\
&\Longrightarrow \\
\mathrm{d}f &= \partial_x f(x) \cdot \mathrm{d}x \text{ on } \mathcal{X}
\end{aligned}
$$

03.02 DERIVATIVE OF A PIECEWISE LINEAR FUNCTION ON A BOUNDED INTERVAL $\mathcal{X} = \cup_{n=1}^{N} \mathcal{X}_n; \mathcal{X}_n = \{x \in \mathbb{R}; x_{n-1} \le x \le x_n; n = 1, \ldots, N\}$

$$
\begin{aligned}
f(x) &= \frac{x_n - x}{x_n - x_{n-1}} \cdot f(x_{n-1}) + \frac{x - x_{n-1}}{x_n - x_{n-1}} \cdot f(x_n) \text{ for } x \in \mathcal{X}_n \ (x_{n-1} \le x \le x_n) \\
&\Longrightarrow \\
f(x_n) - f(x_{n-1}) &= \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \cdot (x_n - x_{n-1}) \\
\mathrm{d}f_n &= f(x_n) - f(x_{n-1}) = \text{ differential of } f \text{ on } \mathcal{X}_n \\
\mathrm{d}x_n &= x_n - x_{n-1} = \text{ differential of } x \text{ on } \mathcal{X}_n \\
\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} &= \partial_x f(x)\big|_n = \text{ (first-order) derivative of } f \text{ on } \mathcal{X}_n \\
&\Longrightarrow \\
\mathrm{d}f_n &= \partial_x f(x)\big|_n \cdot \mathrm{d}x_n \text{ on } \mathcal{X}_n
\end{aligned}
$$

03.03 DERIVATIVE OF A DIFFERENTIABLE FUNCTION ON A BOUNDED INTERVAL $\mathcal{X}$

If for any $x \in \mathcal{X}$

$$\bullet\ \partial_x f^- = \lim_{\Delta x \uparrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ left derivative at } x$$

and

$$\bullet\ \partial_x f^+ = \lim_{\Delta x \downarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \text{ right derivative at } x$$

exist, and

$$\bullet\ \partial_x f^- = \partial_x f^+$$

f is *differentiable at* x and

$$\bullet\ \partial_x f^- = \partial_x f^+ = \partial_x f \text{ (first-order) derivative of } f \text{ at } x$$

03.04 LOCAL DIFFERENTIAL OF A DIFFERENTIABLE FUNCTION

If $f(x)$ is differentiable on $\mathcal{X}$ then

$$\begin{aligned} \mathrm{d}f &= \partial_x f \cdot \mathrm{d}x \\ \mathrm{d}f &= \text{ differential of } f \text{ at } x \\ \mathrm{d}x &= \text{ differential of } x \text{ at } x \\ \partial_x f &= \text{ (first-order) derivative of } f \text{ at } x \end{aligned}$$

03.05 HIGHER-ORDER DERIVATIVES

$$\begin{aligned} \partial_x^0 f(x) &= f(x) \\ \partial_x^n f(x) &= \partial_x[\partial_x^{n-1} f(x)] \text{ for } n = 1, \ldots \\ \bullet\ \partial_x^n f(x) &= n\text{-th order derivative of } f \text{ at } x \end{aligned}$$

$\Longrightarrow$ 04 MATHEMATICAL TOOLS (INTEGRAL CALCULUS) $\Longleftarrow$

04.01 INTEGRAL OF A LINEAR FUNCTION OVER A BOUNDED INTERVAL

$\mathcal{X} : \{x \in \mathbb{R}; x_0 \le x \le x_1\}$

$$f(x) = \frac{x_1 - x}{x_1 - x_0} \cdot f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot f(x_1) \text{ for } x \in \mathcal{X} \; (x_0 \le x \le x_1)$$

$\Longrightarrow$

- $\frac{1}{2}[f(x_1) + f(x_0)] \cdot (x_1 - x_0) = \int_{x=x_0}^{x_1} f(x)\mathrm{d}x =$ integral of f over $\mathcal{X}$ (area of trapezoid)

04.02 INTEGRAL OF A PIECEWISE LINEAR FUNCTION OVER A BOUNDED INTERVAL $\mathcal{X} = \cup_{n=1}^{N} \mathcal{X}_n; \mathcal{X}_n = \{x \in \mathbb{R}; x_{n-1} \le x \le x_n; n = 1, \ldots, N\}$

$$f(x) = \frac{x_n - x}{x_n - x_{n-1}} \cdot f(x_{n-1}) + \frac{x - x_{n-1}}{x_n - x_{n-1}} \cdot f(x_n) \text{ for } x \in \mathcal{X}_n \; (x_{n-1} \le x \le x_n)$$

- $\int_{x=x_0}^{x_N} f(x)\mathrm{d}x = \sum_{n=1}^{N} \int_{x=x_{n-1}}^{x_n} f(x)\mathrm{d}x = \sum_{n=1}^{N} \frac{1}{2}[f(x_n) + f(x_{n-1})] \cdot (x_x - x_{n-1})$ (trapezoidal integration rule)

04.03 INTEGRAL OF AN INTEGRABLE FUNCTION OVER A BOUNDED INTERVAL $\mathcal{X}$

If for all piecewise linear approximations to $f(x)$ defined on the bounded interval $\mathcal{X} = \cup_{n=1}^{N} \mathcal{X}_n; \mathcal{X}_n = \{x \in \mathbb{R}; x_{n-1} \le x \le x_n; n = 1, \ldots, N\}$

- $\int_{\mathcal{X}} f(x)\mathrm{d}x = \lim_{N \to \infty, x_n - x_{n-1} \downarrow 0} \sum_{n=1}^{N} \frac{1}{2}[f(x_n) + f(x_{n-1})] \cdot (x_x - x_{n-1})$

exists, then $\int_{\mathcal{X}} f(x)\mathrm{d}x$ is the *integral of* $f(x)$ *over the interval* $\mathcal{X}$.
Property:

- $\left| \int_{x=a}^{b} f(x)\mathrm{d}x \right| \le \int_{x=a}^{b} |f(x)|\,\mathrm{d}x \le |b - a| \cdot \max_{\{a \le x \le b\}} |f(x)|$

04.04 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS FOR PIECEWISE LINEAR FUNCTION ON BOUNDED INTERVAL $\mathcal{X} = \cup_{n=1}^{N} \mathcal{X}_n; \mathcal{X}_n = \{x \in \mathbb{R}; x_{n-1} \le x \le x_n; n = 1, \dots, N\}$

For any piecewise linear function defined on $\mathcal{X}$

$$f(x) = \frac{x_n - x}{x_n - x_{n-1}} \cdot f(x_{n-1}) + \frac{x - x_{n-1}}{x_n - x_{n-1}} \cdot f(x_n) \text{ for } x \in \mathcal{X}_n \ (x_{n-1} \le x \le x_n)$$

we have

$$\partial_x f(x) = \partial_x f(x)|_n = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \text{ for } x \in \mathcal{X}_n$$

Hence,

- $$\int_{x=x_0}^{x_N} \partial_x f(x) \mathrm{d}x = \sum_{n=1}^{N} \int_{x=x_{n-1}}^{x_n} \partial_x f(x)|_n \, \mathrm{d}x = \sum_{n=1}^{N} \partial_x f(x)|_n \cdot (x_n - x_{n-1})$$
$$= \sum_{n=1}^{N} \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \cdot (x_n - x_{n-1}) = \sum_{n=1}^{N} [f(x_n) - f(x_{n-1})]$$
$$= f(x_N) - f(x_0)$$

04.05 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS FOR FUNCTIONS WITH INTEGRABLE DERIVATIVE ON BOUNDED INTERVAL $\mathcal{X}$

For any function $f(x)$ defined on the bounded interval $\mathcal{X} = \{x \in \mathbb{R}; a \le x \le b\}$, with integrable derivative $\partial_x f(x)$ on $\mathcal{X}$,

- $$\int_{x=a}^{b} \partial_x f(x) \mathrm{d}x = f(b) - f(a) = f(x)\Big|_{x=a}^{x=b}$$

⟹ 05 MATHEMATICAL TOOLS (MISCELLANEOUS) ⟸

05.01 BINOMIAL THEOREM

Binomial expansion ($\{B_n^N; n = 0, \ldots, N\}$ = sequence of binomial coefficients):

- $$(x+y)^N = \sum_{n=0}^{N} B_n^N \cdot x^{N-n} \cdot y^n$$

Evidently,

$$B_0^0 = 1$$
$$B_0^1 = 1 \qquad B_1^1 = 1$$

Recursion formula:

$$(x+y)^{N+1} = (x+y)\cdot(x+y)^N$$

$$\Longrightarrow$$

$$\begin{aligned}
\sum_{n=0}^{N+1} B_n^{N+1} \cdot x^{N+1-n} \cdot y^n &= (x+y) \cdot \sum_{n=0}^{N} B_n^N \cdot x^{N-n} \cdot y^n \\
&= \sum_{n=0}^{N} B_n^N \cdot [x^{N-n} \cdot y^{n+1} + x^{N+1-n} \cdot y^n] \\
&= \sum_{m=1}^{N+1} B_{m-1}^N \cdot x^{N+1-m} \cdot y^m + \sum_{n=0}^{N} B_n^N \cdot x^{N-n+1} \cdot y^n
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
B_0^{N+1} &= B_0^N = 1 \\
B_{N+1}^{N+1} &= B_N^N = 1 \\
B_n^{N+1} &= B_{n-1}^N + B_n^N \text{ for } n = 1, \ldots, N
\end{aligned}$$

- 'Pascal's' triangle:

'PASCAL'S' TRIANGLE

N	B_n^N
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

05.02 DERIVATIVE OF THE FUNCTION $f(x) = x^N; N = 0, 1, 2, \ldots$

Derivative of $f(x) = x^N; N = 0, 1, 2, \ldots$;:

$$\begin{aligned}
\partial_x f(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^N - x^N}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sum_{n=0}^{N} B_n^N \cdot x^{N-n} \cdot (\Delta x)^n - x^N}{\Delta x} \\
&= B_1^N \cdot x^{N-1} \\
&= N \cdot x^{N-1} \\
&\Longrightarrow \\
\bullet\ \partial_x x^N &= N \cdot x^{N-1} \text{ for } N = 0, 1, 2, \ldots
\end{aligned}$$

05.03 DERIVATIVE OF THE PRODUCT OF TWO LINEAR FUNCTIONS ON A BOUNDED INTERVAL $\mathcal{X} : \{x \in \mathbb{R}; x_0 \le x \le x_1\}$

$$\begin{aligned}
f(x) &= \frac{x_1 - x}{x_1 - x_0} \cdot f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot f(x_1) \text{ for } x \in \mathcal{X}\ (x_0 \le x \le x_1) \\
&\Longrightarrow \\
\partial_x f(x) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
g(x) &= \frac{x_1 - x}{x_1 - x_0} \cdot g(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot g(x_1) \text{ for } x \in \mathcal{X}\ (x_0 \le x \le x_1) \\
&\Longrightarrow \\
\partial_x g(x) &= \frac{g(x_1) - g(x_0)}{x_1 - x_0} \\
&\Longrightarrow \text{(via } \partial_x (x - x_0)^2 = 2 \cdot (x - x_0)) : \\
\bullet\ \partial_x [f(x) \cdot g(x)] &= [\partial_x f(x)] \cdot g(x) + f(x) \cdot [\partial_x g(x)]
\end{aligned}$$

05.04 PRODUCT RULE IN DIFFERENTIATION ON A BOUNDED INTERVAL

BOUNDED INTERVAL $\mathcal{X}$

$$\bullet\ \partial_x [f(x) \cdot g(x)] = [\partial_x f(x)] \cdot g(x) + f(x) \cdot [\partial_x g(x)] \text{ for } x \in \mathcal{X}$$

05.05 INTEGRATION BY PARTS ON A BOUNDED INTERVAL

$\mathcal{X} = \{x \in \mathbb{R}; a \leq x \leq b\}$

- $$\int_{x=a}^{b} \{[\partial_x f(x)] \cdot g(x) + f(x) \cdot [\partial_x g(x)]\} \mathrm{d}x = \int_{x=a}^{b} \partial_x [f(x) \cdot g(x)]$$
$$= f(b) \cdot g(b) - f(a) \cdot g(a) = f(x) \cdot g(x) \Big|_{x=a}^{x=b}$$

Note: Often this result is applied in the form:

- $$\int_{x=a}^{b} \{[\partial_x f(x)] \cdot g(x) = f(x) \cdot g(x) \Big|_{x=a}^{x=b} - \int_{x=a}^{b} f(x) \cdot [\partial_x g(x)]\} \mathrm{d}x$$

05.06 TAYLOR EXPANSION WITH GLOBAL REMAINDER (FUNCTION OF A SINGLE VARIABLE)

Note: Use 'Taylor's trick' • $\dfrac{(x-\xi)^n}{n!} = -\partial_\xi \left[\dfrac{(x-\xi)^{n+1}}{(n+1)!}\right]$ for $n = 0, 1, 2, \ldots$ in:

$$
\begin{aligned}
f(x) - f(a) &= \int_{\xi=a}^{x} \partial_\xi f(\xi)\mathrm{d}\xi \\
&= -\int_{\xi=a}^{x} \partial_\xi f(\xi)\partial_\xi \left[\frac{(x-\xi)}{1!}\right] \mathrm{d}\xi) \\
&= \left[-\partial_\xi f(\xi) \cdot \left[\frac{(x-\xi)}{1!}\right]\right]\Bigg|_{\xi=a}^{x} + \int_{\xi=a}^{x} \partial_\xi^2 f(\xi) \cdot \left[\frac{(x-\xi)}{1!}\right] \mathrm{d}\xi \\
&= [\partial_\xi f(\xi)]\big|_{\xi=a} \cdot \frac{(x-a)}{1!} - \int_{\xi=a}^{x} \partial_\xi^2 f(\xi) \cdot \partial_\xi \left[\frac{(x-\xi)^2}{(2)!}\right] \mathrm{d}\xi
\end{aligned}
$$

$\Longrightarrow$

$$
\bullet\ f(x) = f(a) + \sum_{n=1}^{N-1} [\partial_\xi^n f(\xi)]\big|_{\xi=a} \cdot \frac{(x-a)^n}{n!} + \int_{\xi=a}^{x} \partial_\xi^N f(\xi) \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi
$$

where the *'(global) remainder after N terms'* (counting from $N = 0$) is

$$
\bullet\ R_N = \int_{\xi=a}^{x} \partial_\xi^N f(\xi) \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi
$$

Let

$$
\partial_\xi^N f(\xi) \leq M \text{ for } a \leq \xi \leq x
$$

then

$$
\begin{aligned}
|R_N| &\leq M \cdot \left| \int_{\xi=a}^{x} \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi \right| \\
&= M \cdot \frac{|x-a|^N}{N!}
\end{aligned}
$$

or, equivalently,

$$
\bullet\ R_N = O[(x-a)^N] \text{ for all } x \in \mathbb{R}
$$

05.07 TAYLOR EXPANSION WITH LOCAL REMAINDER (FUNCTION OF A SINGLE VARIABLE)

Note: Use 'Taylor's trick' • $\dfrac{(x-\xi)^n}{n!} = -\partial_\xi \left[\dfrac{(x-\xi)^{n+1}}{(n+1)!}\right]$ for $n = 0, 1, 2, \ldots$ in:

$$
\begin{aligned}
f(x) - f(a) &= \int_{\xi=a}^{x} \partial_\xi f(\xi) \mathrm{d}\xi \\
&= -\int_{\xi=a}^{x} \partial_\xi f(\xi) \partial_\xi \left[\frac{(x-\xi)}{1!}\right] \mathrm{d}\xi) \\
&= \left[-\partial_\xi f(\xi) \cdot \left[\frac{(x-\xi)}{1!}\right]\right]\Bigg|_{\xi=a}^{x} + \int_{\xi=a}^{x} \partial_\xi^2 f(\xi) \cdot \frac{(x-\xi)}{1!} \mathrm{d}\xi \\
&= [\partial_\xi f(\xi)]|_{\xi=a} \cdot \frac{(x-a)}{1!} + \int_{\xi=a}^{x} \partial_\xi^2 f(\xi) \cdot \partial_\xi \left[\frac{(x-\xi)^2}{(2)!}\right] \mathrm{d}\xi \\
&\Longrightarrow \\
\bullet\ f(x) &= f(a) + \sum_{n=1}^{N-1} [\partial_\xi^n f(\xi)]|_{\xi=a} \cdot \frac{(x-a)^n}{n!} + \int_{\xi=a}^{x} \partial_\xi^N f(\xi) \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi
\end{aligned}
$$

However,

$$
\begin{aligned}
\int_{\xi=a}^{x} \partial_\xi^N f(\xi) \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi &= [\partial_\xi^n f(\xi)]|_{\xi=a} \cdot \int_{\xi=a}^{x} \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi + \\
&\qquad \int_{\xi=a}^{x} \{\partial_\xi^N f(\xi) - [\partial_\xi^n f(\xi)]|_{\xi=a}\} \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi \\
&= [\partial_\xi^n f(\xi)]|_{\xi=a} \cdot \frac{(x-a)^N}{N!} + r_N \\
\bullet\ r_N &= \int_{\xi=a}^{x} \{\partial_\xi^N f(\xi) - [\partial_\xi^N f(\xi)]|_{\xi=a}\} \cdot \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi
\end{aligned}
$$

In view of the assumed continuity of $\partial_\xi^N f(\xi)$ over the interval $a \le \xi \le x$ we have (see Section 04.03)

$$\left|\partial_\xi^N f(\xi) - [\partial_\xi^N f(\xi)]|_{\xi=a}\right| \le m(|x-a|) \text{ where } m(|x-a|) \to 0 \text{ as } x \to a$$

whence for the *'(local) remainder after $N+1$ terms'* (counting from $N = 0$) we have

$$
\begin{aligned}
|r_N| &\le m(|x-a|) \cdot \left| \int_{\xi=a}^{x} \frac{(x-\xi)^{N-1}}{(N-1)!} \mathrm{d}\xi \right| \\
&= m(|x-a|) \cdot \frac{|x-a|^N}{N!}
\end{aligned}
$$

or, equivalently,

$$\bullet\ r_N = o[(x-a)^N] \text{ as } x \to a$$

05.08 CHAIN RULE FOR DIFFERENTIATION

Let $f(g)$ be a differentiable function of g and $g(x)$ be a differentiable function of x on some interval $\{x \in \mathcal{D}_{\rm var} \subset \mathbb{R}\}$. Then Taylor's expansion gives

$$g(x+\Delta x)) = g(x) + \partial_x g(x) \cdot \frac{1}{1!}\Delta x + o(\Delta x) \text{ as } \Delta x \to 0$$
$$f(g+\Delta g)) = f(g) + \partial_g f(g) \cdot \frac{1}{1!}\Delta g + o(\Delta g) \text{ as } \Delta g \to 0$$

where

$$\begin{aligned} \Delta g &= g(x+\Delta x) - g(x) \\ &\Longrightarrow \left(\text{via } \frac{1}{1!} = 1,\ o(\Delta g) = o(\Delta x)\right) : \\ f[g(x+\Delta x)] &= f[g(x)] + \partial_g f(g) \cdot \partial_x g(x) \cdot \Delta x + o(\Delta x) \end{aligned}$$

Using the definition of derivative, it follows that (*'chain rule for differentiation'*)

$$\bullet\ \partial_x f[g(x)] = \lim_{\Delta x \to 0} \frac{f[g(x+\Delta x)] - f[g(x)]}{\Delta x}$$
$$= \partial_g f(g) \cdot \partial_x g(x) \text{ for } x \in \mathcal{D}_{\rm var}$$

05.09a TAYLOR EXPANSION (FUNCTION OF TWO VARIABLES)

Consider the function of two variables $f(x, y)$. To construct the Taylor expansion of $f(x, y)$ about $x = x_0, y = y_0$, we set $x = x_0 + \xi \cdot t, y = y_0 + \eta \cdot t$, in which ξ and η are fixed, subject to the condition $\xi^2 + \eta^2 = 1$, and t is the (single) variable into which the function $f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)$ is Taylor expanded about $t = 0$. The result is:

- $$f(x_0 + \xi \cdot t, y_0 + \eta \cdot t) = f(x_0, y_0) + \sum_{n=1}^{N} \partial_t^n f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} \cdot \frac{t^n}{n!} + O(t^{N+1})$$

(global remainder)

or

$$f(x_0 + \xi \cdot t, y_0 + \eta \cdot t) = f(x_0, y_0) + \sum_{n=1}^{N} \partial_t^n f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} \cdot \frac{t^n}{n!} + 0(t^{N})$$

(local remainder)

Now,

$$\partial_t f(x_0 + \xi \cdot t, y_0 + \eta \cdot t) = \xi \cdot \partial_x f(x_0 + \xi \cdot t, y_0 + \eta \cdot t) + \eta \cdot \partial_y f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)$$

and hence,

$$\begin{aligned}
&\partial_t f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} = \\
&\xi \cdot \partial_x f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} + \\
&\eta \cdot \partial_y f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} = \\
&\xi \cdot \partial_x f(x, y)\big|_{x=x_0, y=y_0} + \eta \cdot \partial_y f(x, y)\big|_{x=x_0, y=y_0} = \\
&(\xi \cdot \partial_x + \eta \cdot \partial_y) f(x, y)\big|_{x=x_0, y=y_0}
\end{aligned}$$

Consequently, using Newton's binomial theorem,

$$\begin{aligned}
&\partial_t^n f(x_0 + \xi \cdot t, y_0 + \eta \cdot t)\big|_{t=0} = \\
&(\xi \cdot \partial_x + \eta \cdot \partial_y)^n f(x, y)\big|_{x=x_0, y=y_0} = \\
&\left[\sum_{p=1}^{n} B_p^n \cdot (\xi \cdot \partial_x)^{n-p} \cdot (\eta \cdot \partial_y)^p)\right] f(x, y)\big|_{x=x_0, y=y_0}
\end{aligned}$$

Continued in Section 05.09b

05.09b TAYLOR EXPANSION (FUNCTION OF TWO VARIABLES) (CONT'D)

With $(\xi^{n-p}\cdot\eta^p)\cdot t^n = (x-x_0)^{n-p}\cdot(y-y_0)^p$ and $d(x-x_0, y-y_0) = [(x-x_0)^2+(y-y_0)^2]^{1/2} > 0$ as the Euclidean distance from $\{x_0, y_0\}$ to $\{x, y\}$, the final result can be written as

- $f(x,y) = f(x_0,y_0) +$
  $\sum_{n=1}^{N} \frac{1}{n!} \cdot \left[\sum_{p=1}^{n} B_p^n \cdot (x-x_0)^{n-p} \cdot (y-y_0)^p \partial_x^{n-p}\partial_y^p f(x,y)\big|_{x=x_0,y=y_0} \right]$
  $+O[d(x-x_0, y-y_0)^{N+1}]$ (global remainder)

or

- $f(x,y) = f(x_0,y_0) +$
  $\sum_{n=1}^{N} \frac{1}{n!} \cdot \left[\sum_{p=1}^{n} B_p^n \cdot (x-x_0)^{n-p} \cdot (y-y_0)^p \partial_x^{n-p}\partial_y^p f(x,y)\big|_{x=x_0,y=y_0} \right]$
  $+o[d(x-x_0, y-y_0)^{N}]$ as $d(x-x_0, y-y_0) \to 0$ (local remainder)

05.10 TAYLOR EXPANSION (EXERCISES)

Exercise: The twice differentiable function $f(x)$ is defined on the domain $\mathcal{D}_{\text{var}} \subset \mathbb{R}$. Show that its Taylor expansion with local remainder about $x_0 \in \mathcal{D}_{\text{var}}$ is given by

- $f(x) = f(x_0) + \partial_x f(x)\big|_{x=x_0} \cdot (x-x_0) + \frac{1}{2} \cdot \partial_x^2 f(x)\big|_{x=x_0} \cdot (x-x_0)^2 +$
  $o(|x-x_0|)$ as $x \to x_0$

Exercise: The twice differentiable function $f(x,y)$ is defined on the domain $\mathcal{D}_{\text{var}} \subset \mathbb{R}^2$. Show that its Taylor expansion with local remainder about $(x_0, y_0) \in \mathcal{D}_{\text{var}}$ is given by

- $f(x,y) = f(x_0,y_0) + \partial_x f(x,y)\big|_{x=x_0,y=y_0} \cdot (x-x_0) + \partial_y f(x,y)\big|_{x=x_0,y=y_0} \cdot (y-y_0) +$
  $\frac{1}{2} \cdot \partial_x^2 f(x,y)\big|_{x=x_0,y=y0} \cdot (x-x_0)^2 +$
  $\partial_x\partial_y f(x,y)\big|_{x=x_0,y=y0} \cdot (x-x_0) \cdot (y-y_0) +$
  $\frac{1}{2} \cdot \partial_y^2 f(x,y)\big|_{x=x_0,y=y0} \cdot (y-y_0)^2 +$
  $o[d(x-x_0, y-y_0)]$ as $x \to x_0$

05.11 SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

(GAUSSIAN ELIMINATION)

The method of *Gaussian elimination* to solve *systems of linear algebraic equations* consists of two parts:

- rearrange the system into an *upper triangular form*
- solve for the unknowns by *back substitution*

Rearrangement into upper triangular form
Starting system:

$$a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + a_{1,3} \cdot x_3 = u_1 \quad (1)$$
$$a_{2,1} \cdot x_1 + a_{2,2} \cdot x_2 + a_{2,3} \cdot x_3 = u_2 \quad (2)$$
$$a_{3,1} \cdot x_1 + a_{3,2} \cdot x_2 + a_{3,3} \cdot x_3 = u_3 \quad (3)$$

First step:

$$a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + a_{1,3} \cdot x_3 = u_1 \quad (4)$$
$$a_{2,1} \cdot \text{Eq.}(1) - a_{1,1} \cdot \text{Eq.}(2) \Longrightarrow \quad 0 + b_{2,2} \cdot x_2 + b_{2,3} \cdot x_3 = v_2 \quad (5)$$
$$a_{3,1} \cdot \text{Eq.}(1) - a_{1,1} \cdot \text{Eq.}(3) \Longrightarrow \quad 0 + b_{3,2} \cdot x_2 + b_{3,3} \cdot x_3 = v_3 \quad (6)$$

. . .
Final step:

$$a_{1,1} \cdot x_1 + a_{1,2} \cdot x_2 + a_{1,3} \cdot x_3 = u_1 \quad (7)$$
$$0 + b_{2,2} \cdot x_2 + b_{2,3} \cdot x_3 = v_2 \quad (8)$$
$$b_{3,2} \cdot \text{Eq.}(6) - b_{2,2} \cdot \text{Eq.}(5) \Longrightarrow \quad 0 + 0 + c_{3,3} \cdot x_3 = w_3 \quad (9)$$

Back substitution:

$$\text{Eq.}(9) \Longrightarrow \quad x_3 = \frac{1}{c_{3,3}} \cdot w_3 \quad (10)$$
$$\text{Eq.}(8) \Longrightarrow \quad x_2 = \frac{1}{b_{2,2}} \cdot v_2 - \frac{b_{2,3}}{b_{2,2}} \cdot x_3 \quad (11)$$
$$\text{Eq.}(7) \Longrightarrow \quad x_1 = \frac{1}{a_{1,1}} \cdot u_1 - \frac{a_{1,2}}{a_{1,1}} \cdot x_2 - \frac{a_{1,3}}{a_{1,1}} \cdot x_3 \quad (12)$$

$\Longrightarrow$ 06 SPECIAL FUNCTIONS $\Longleftarrow$

06.01 EXPONENTIAL FUNCTION

The *'exponential function'* $f(x) = \exp(x)$ is defined through the relations

$$\bullet \ \partial_x f(x) = f(x) \ \text{ and } \ \bullet \ f(0) = 1$$

Evidently,

$$\partial_x^n f(x) = f(x) \Longrightarrow \partial_x^n f(x)\big|_{x=0} = 1$$

Taylor series (Taylor expansion can be proved to converge as $N \to \infty$):

$$\bullet \ \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for all } x \in \mathbb{R}$$

06.02 EXPONENTIAL FUNCTION (EXERCISES)

Exercise: Show that $f(x) = A \cdot \exp(B \cdot x)$ satisfies the relations $\partial_x f(x) = B \cdot f(x)$ and $f(0) = A$. (*Hint:* Apply the chain rule for differentiation.)

Exercise: Show that $f(x) = \exp(a + x)$ satisfies the relations $\partial_x f(x) = f(x)$ and $f(0) = \exp(a)$. (*Hint:* Apply the chain rule for differentiation.) Corollary:

$$\bullet \ \exp(a + x) = \exp(a) \cdot \exp(x)$$

Exercise: Show that $f(x) = B \cdot \exp[\int_{\xi=0}^{x} \phi(\xi) \mathrm{d}\xi]$ satisfies the relations $\partial_x f(x) = \phi(x) \cdot f(x)$ and $f(0) = B$. (*Hint:* Apply the chain rule for differentiation and the main theorem of integral calculus.)
Exercise:

$$\bullet \ \exp(-x) < \frac{N!}{x^N} \text{ for any } N = 0, 1, 2, \ldots \text{ and } x > 0$$

(*Hint:* Use the property $\exp(-x) = 1/\exp(x)$ and the Taylor expansion for $\exp(x)$.) Corollary:

$$\bullet \ x^N \cdot \exp(-x) \to 0 \text{ as } x \to \infty \text{ for any } N = 0, 1, 2, \ldots$$

06.03 COSINE AND SINE FUNCTIONS

The *'cosine function'* $f(x) = \cos(x)$ and the *'sine function* $g(x) = \sin(x)$ are defined through the relations

- $\partial_x f(x) = -g(x)$; • $\partial_x g(x) = f(x)$; • $f(0) = 1,;$ • $g(0) = 0$

Evidently,

$$\partial_x^2 f(x) = -f(x), \quad \partial_x^2 g(x) = -g(x)$$
$$\partial_x^{2n} f(x)\big|_{x=0} = (-1)^n, \quad \partial_x^{2n+1} f(x)\big|_{x=0} = 0 \text{ for } n = 0, 1, 2, \ldots$$
$$\partial_x^{2n+1} g(x)\big|_{x=0} = (-1)^n, \quad \partial_x^{2n} g(x)\big|_{x=0} = 0, \text{ for } n = 0, 1, 2, \ldots$$

Taylor series (Taylor expansion can be proved to converge as $N \to \infty$):

- $\cos(x) = \sum_{n=0}^{\infty} (-1)^{2n} \cdot \frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^{2n+1} \cdot \frac{x^{2n+1}}{(2n+1)!}$ for all $x \in \mathbb{R}$

06.04 COSINE AND SINE FUNCTIONS (EXERCISES)

Exercise: Show that • $\cos(-x) = \cos(x)$ and • $\sin(-x) = -\sin(x)$

Exercise: Show that $f(x) = \cos(a + x)$ and $g(x) = \sin(a + x)$ satisfy the relations $\partial_x f(x) = -g(x)$ and $\partial_x g(x) = f(x)$, together with $f(0) = \cos(a)$ and $g(0) = \sin(a)$. (*Hint:* Apply the chain rule for differentiation.) Show that these relations lead to

- $\cos(a + x) = \cos(a) \cdot \cos(x) - \sin(a) \cdot \sin(x)$
- $\sin(a + x) = \sin(a) \cdot \cos(x) + \cos(a) \cdot \sin(x)$

Exercise: Let

- $f(x) = \cos^2(x) + \sin^2(x)$

Show that

$$\partial_x f(x) = 0$$

and conclude that

- $\cos^2(x) + \sin^2(x) = 1$

06.05 (NATURAL) LOGARITHM

The *'(natural) logarithmic function'* $u = \ln(x)$ is (implicitly) defined through the relation

$$\exp(u) = x \text{ for } x \in \mathbb{R}, x > 0$$

i.e.,

$$\bullet\ \exp[\ln(x)] = x \text{ for } x \in \mathbb{R}, x > 0$$

Properties:

$$\begin{aligned}
\exp[\ln(1)] = 1 = \exp(0) &\Longrightarrow \ln(1) = 0 \\
\exp[\ln(e)] = e = \exp(1) &\Longrightarrow \ln(e) = 1 \\
\exp[\ln(x \cdot y)] = x \cdot y = & \\
\exp[\ln(x)] \cdot \exp[\ln(y)] = \exp[\ln(x) + \ln(y)] &\Longrightarrow \bullet\ \ln(x \cdot y) = \ln(x) + \ln(y) \\
\partial_x \exp[\ln(x)] = 1 &\Longrightarrow \exp[\ln(x)] \cdot \partial_x \ln(x) = 1 \\
&\Longrightarrow x \cdot \partial_x \ln(x) = 1 \\
&\Longrightarrow \partial_x \ln(x) = \frac{1}{x} \text{ for } x \in \mathbb{R}, x > 0 \\
\exp\left[\ln\left(\frac{1}{x}\right)\right] = \frac{1}{x} = \frac{1}{\exp[\ln(x)]} = \exp[-\ln(x)] &\Longrightarrow \ln\left(\frac{1}{x}\right) = -\ln(x) \\
\exp[\ln(x^p)] = x^p = \{\exp[\ln(x)]\}^p = \exp[p \cdot \ln(x)] &\Longrightarrow \bullet\ \ln(x^p) = p \cdot \ln(x)
\end{aligned}$$

The number $e = \exp(1)$

$$\begin{aligned}
\partial_x[\ln(x)]\big|_{x=1} &= \lim_{h \to 0} \frac{\ln(x+h) - \ln(1)}{h} \\
&= \lim_{h \to 0} \ln\left[(1+h)^{1/h}\right] \\
&= \ln\left[\lim_{h \to 0} (1+h)^{1/h}\right]
\end{aligned}$$

or

$$1 = \ln\left[\lim_{h \to 0} (1+h)^{1/h}\right]$$

$\Longrightarrow$

$$\lim_{h \to 0} (1+h)^{1/h} = \exp(1) = e$$

06.06 (NATURAL) LOGARITHM (EXERCISES)

Exercise: Show that

- $\ln\left(\frac{x^p}{y^q}\right) = p \cdot \ln(x) - q \cdot \ln(y)$ for $x \in \mathbb{R}, x > 0$ and $y \in \mathbb{R}, y > 0$

Exercise: Show that

- $\ln(x) = \int_{\xi=1}^{x} \frac{1}{\xi} \mathrm{d}\xi$ for $x \in \mathbb{R}, x > 0$

(*Hint:* Use the property $1/\xi = \partial_\xi \ln(\xi)$ and the main theorem of integral calculus.)

Exercise: Show that the Taylor expansion of $f(x) = \ln(1-x)$ is given by

- $\ln(1-x) = -\sum_{n=1}^{N} \frac{x^n}{n} + O(x^{N+1})$ (global remainder)

or

- $\ln(1-x) = -\sum_{n=1}^{N} \frac{x^n}{n} + o(x^N)$ as $x \to 0$ (local remainder)

(*Hint:* Observe that $f(0) = 0$ and that $\partial_x^n f(x) = -(n-1)!/(1-x)^n$ for $n = 1, 2, 3, \ldots$.)
Note: The expansion can be shown to converge as $N \to \infty$ for $|x| < 1$.

06.07 DOUBLE LOGARITHMIC DERIVATIVE, (ECONOMIC) ELASTICITY

For 'wildly' varying functions, the graph is often plotted on a *logarithmic scale*, i.e., the graph of $y = f(x)$ is represented with $\ln(x)$ along the horizontal axis and $\ln(y)$ along the vertical axis. Of course, this can only be done if both $x > 0$ and $y > 0$. In the relevant graph, the local variations are characterized by the variations in slope, i.e., by $\partial_{\ln(x)} \ln(y)$. With

- $x = \exp(u)$
- $y = \exp(v)$

this slope can be rewritten as:

$$\partial_{\ln(x)} \ln(y) = \partial_u v(u)$$

Now, differentiation of

$$\exp(v) = f[\exp(u)]$$

with respect to u yields

$$\exp(v) \cdot \partial_u v(u) = \partial_x f[\exp(u)] \cdot \exp(u)$$

which leads to

- $\partial_u v(u) = \dfrac{x}{f(x)} \cdot \partial_x f(x)$

Note: In Mathematical Economics the resulting right-hand side is used as the definition of the elasticity of $f(x)$ with respect to x:

- $El_x f(x) = \dfrac{x}{f(x)} \cdot \partial_x f(x)$

The latter definition makes no reference to the logarithmic scale and therefore the quantities x and $f(x)$ need not be positive!

06.08 LAPLACE INTEGRALS

Integrals of the type

$$\bullet\ \mathbf{L}_f(s) = \int_{t=0}^{\infty} \exp(-s \cdot t) \cdot f(t)\mathrm{d}t$$

where $f(t)$ is an integrable function of t on the interval $\{t \in \mathbb{R}, 0 < t < \infty\}$ and $s \in \mathbb{R}, s > 0$ are known as *Laplace integrals*. For

$$f(t) = \frac{t^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

they can easily be evaluated. Note in this respect that

$$\begin{aligned}\int_{t=0}^{\infty} \exp(-s \cdot t) \cdot \frac{t^n}{n!}\mathrm{d}t &= -\frac{1}{s} \cdot \int_{t=0}^{\infty} \frac{t^n}{n!}\mathrm{d}[\exp(-s \cdot t)] \\ &= -\frac{1}{s} \cdot \left[\frac{t^n}{n!} \cdot \exp(-s \cdot t)\right]\Bigg|_{t=0}^{\infty} + \\ &\quad \frac{1}{s} \cdot \int_{t=0}^{\infty} \exp(-s \cdot t) \cdot \frac{t^{n-1}}{(n-1)!}\mathrm{d}t \\ &= \frac{1}{s} \cdot \int_{t=0}^{\infty} \exp(-s \cdot t) \cdot \frac{t^{n-1}}{(n-1)!}\mathrm{d}t \text{ for } s > 0, n = 1, 2, 3, \ldots\end{aligned}$$

while

$$\int_{t=0}^{\infty} \exp(-s \cdot t)\mathrm{d}t = \frac{1}{s} \text{ for } s > 0$$

Consequently,

$$\int_{t=0}^{\infty} \exp(-s \cdot t) \cdot \frac{t^n}{n!}\mathrm{d}t = \frac{1}{s^{n+1}} \text{ for } s > 0, n = 0, 1, 2, \ldots$$

$\Longrightarrow$ 07 OPTIMIZATION TECHNIQUES $\Longleftarrow$

07.01 INTERIOR AND BOUNDARY EXTREMA OF TARGET FUNCTION

A differentiable economical target function defined on the bounded domain $\mathcal{D}_{\rm var}$ can have

- interior extrema (i.e., extrema (maxima or minima) at a number of interior points of $\mathcal{D}_{\rm var}$)
- boundary extrema (i.e., extrema (maxima or minima) at a number of points on the boundary $\partial\mathcal{D}_{\rm var}$ of $\mathcal{D}_{\rm var}$)

Extrema belong to the set of *stationary points* of the target function, i.e.,the points that are found upon putting equal to zero the first derivative(s) of the target function. Stationary points on $\partial\mathcal{D}_{\rm var}$ are found upon putting equal to zero the first derivative(s) of the target function, subject to the *constraint* that the points are points on $\partial\mathcal{D}_{\rm var}$

A stationary point is a *minimum* if in its entire neighborhood the function value is larger than that in the stationary point. A stationary point is a *maximum* if in its entire neighborhood the function value is smaller than that in the stationary point. If neither of these two criteria holds, the stationary point in neither a minimum or a maximum. Which one of these possibilities is the case, is usually tested via the *Taylor expansion with local remainder*.

07.02 INTERIOR EXTREMA OF AN EXPLICIT TARGET FUNCTION OF A SINGLE VARIABLE

Let $f(x)$ be an explicit, twice differentiable target function defined on $\mathcal{D}_{\rm var}$. Let $x_0 \in \mathcal{D}_{\rm var}$ be an interior stationary point, i.e.,

- $\partial_x f(x)\big|_{x=x_0} = 0$

then the Taylor expansion about x_0 with local remainder becomes

- $f(x) = f(x_0) + \frac{1}{2}\cdot(x-x_0)^2\cdot\partial_x^2 f(x)\big|_{x=x_0} + o(|x-x_0|^2)$ as $x \to x_0$

Evidently, the following conclusion holds:

- $\partial_x^2 f(x)\big|_{x=x_0} > 0 \Longrightarrow f(x_0)$ is a (local) minimum of $f(x)$
- $\partial_x^2 f(x)\big|_{x=x_0} < 0 \Longrightarrow f(x_0)$ is a (local) maximum of $f(x)$

If $\partial_x^2 f(x)\big|_{x=x_0} = 0$, further information is needed to decide upon the nature of $f(x_0)$

07.03 INTERIOR EXTREMA OF AN IMPLICIT TARGET FUNCTION OF A SINGLE VARIABLE

Let $f(x)$ be an implicit, twice differentiable target function defined on $\mathcal{D}_{\rm var}$ via the *target functional equation*

$$\bullet\ \Phi[f(x), x] = 0 \text{ for } x \in \mathcal{D}_{\rm var}$$

where Φ is a twice differentiable *functional* acting on its *arguments* $f(x)$ and x. To arrive at the stationary points of $f(x)$ we differentiate the functional equation with respect to x, put $\partial_x f(x) = 0$ and solve for x. To facilitate the *implicit differentiation* involved, we put $u = f(x)$ and use the chain rule for differentiation to obtain

$$\bullet\ \partial_u \Phi[u, x]\big|_{u=f(x)} \cdot \partial_x f(x) + \partial_x \Phi[u, x]\big|_{u=f(x)} = 0$$

With $\bullet\ \partial_x f(x)\big|_{x=x_0} = 0$, x_0 follows from the system of equations

$$\bullet\ \Phi[u, x]\big|_{u=f(x)} = 0 \text{ for } x = x_0$$
$$\bullet\ \partial_x \Phi[u, x]\big|_{u=f(x)} = 0 \text{ for } x = x_0$$

To extract information about $\partial_x^2 f(x)\big|_{x=x_0}$ the once differentiated expression is differentiated once more to yield

$$\bullet\ \partial_u^2 \Phi[u, x]\big|_{u=f(x)} \cdot [\partial_x f(x)]^2 + \partial_u \Phi[u, x]\big|_{u=f(x)} \cdot \partial_x^2 f(x)] + \partial_u \partial_x \Phi[u, x]\big|_{u=f(x)} \partial_x f(x) + \partial_x^2 \Phi[u, x]\big|_{u=f(x)} = 0$$

with the result that

$$\bullet\ \partial_x^2 f(x)]\big|_{x=x_0} = -\frac{\partial_x^2 \Phi[u, x]\big|_{u=f(x), x=x_0}}{\partial_u \Phi[u, x]\big|_{u=f(x_0), x=x_0}}$$

For the rest as to x_0 being a minimum or a maximum, the same conditions as in Section 07.02 apply.

07.04a INTERIOR EXTREMA OF AN EXPLICIT TARGET FUNCTION OF TWO VARIABLES

Let $f(x,y)$ be an explicit, twice differentiable target function defined on $\mathcal{D}_{\rm var}$. Let $(x_0,y_0) \in \mathcal{D}_{\rm var}$ be an interior stationary point, i.e.,

$$\bullet\ \partial_x f(x,y)\big|_{x=x_0,y=y_0} = 0 \text{ and } \bullet\ \partial_y f(x,y)\big|_{x=x_0,y=y_0} = 0$$

then the Taylor expansion about (x_0,y_0) with local remainder becomes

$$\begin{aligned}\bullet\ f(x,y) = {} & f(x_0,y_0) + \\ & \frac{1}{2}\cdot\Big[(x-x_0)^2\cdot\partial_x^2 f(x,y)\big|_{x=x_0,y=y_0} + \\ & 2\cdot(x-x_0)\cdot(y-y_0)\cdot\partial_x\partial_y f(x,y)\big|_{x=x_0,y=y_0} + \\ & (y-y_0)^2\cdot\partial_y^2 f(x,y)\big|_{x=x_0,y=y_0}\Big] + \\ & o[d(x-x_0,y-y_0)^2] \text{ as } (x,y)\to(x_0,y_0)\end{aligned}$$

Evidently, the following conclusions hold:

- $\partial_x^2 f(x,y)\big|_{x=x_0,y-y_0} > 0 \Longrightarrow$ *necessary* condition for $f(x_0,y-y_0)$ to be a (local) minimum of $f(x,y)$(take $y=y_0$)
- $\partial_y^2 f(x,y)\big|_{x=x_0,y-y_0} > 0 \Longrightarrow$ *necessary* condition for $f(x_0,y-y_0)$ to be a (local) minimum of $f(x,y)$(take $x=x_0$)

and

- $\partial_x^2 f(x,y)\big|_{x=x_0,y-y_0} < 0 \Longrightarrow$ *necessary* condition for $f(x_0,y-y_0)$ to be a (local) maximum of $f(x,y)$(take $y=y_0$)
- $\partial_y^2 f(x,y)\big|_{x=x_0,y-y_0} < 0 \Longrightarrow$ *necessary* condition for $f(x_0,y-y_0)$ to be a (local) maximum of $f(x,y)$(take $x=x_0$)

Continued in Section 07.04b

07.04b INTERIOR EXTREMA OF AN EXPLICIT TARGET FUNCTION OF TWO VARIABLES (CONT'D)

The *necessary* conditions are to be supplemented with the *sufficiency condition*

- $\partial_x^2 f(x,y)\big|_{x=x_0,y-y_0} \cdot \partial_y^2 f(x,y)\big|_{x=x_0,y-y_0} - \left[\partial_x \partial_y f(x,y)\big|_{x=x_0,y=y_0}\right]^2 > 0$

which follows from the consideration that the quadratic Taylor expansion terms can be rewritten as

- $$A \cdot (x-x_0)^2 + 2 \cdot B \cdot (x-x_0) \cdot (y-y_0) + C \cdot (y-y_0)^2 = A \cdot \left\{ \left[(x-x_0) + \frac{B}{A} \cdot (y-y_0) \right]^2 + \left(\frac{C}{A} - \frac{B^2}{A^2} \right) (y-y_0)^2 \right\}$$

If $\partial_x^2 f(x,y)\big|_{x=x_0,y=y_0} = 0$ and $\partial_y^2 f(x,y)\big|_{x=x_0,y=y_0} = 0$, further information is needed to decide upon the nature of $f(x_0)$

07.05a INTERIOR EXTREMA OF AN IMPLICIT TARGET FUNCTION OF TWO VARIABLES

Let $f(x,y)$ be an implicit, twice differentiable target function defined on $\mathcal{D}_{\rm var}$ via the *target functional equation*

$$\bullet\ \Phi[f(x,y),x,y] = 0 \text{ for } x \in \mathcal{D}_{\rm var}$$

where Φ is a twice differentiable *functional* acting on its *arguments* $f(x,y)$, x and y. To arrive at the stationary points of $f(x,y)$ we differentiate the functional equation with respect to x and y, put $\partial_x f(x,y) = 0$ and $\partial_y f(x,y) = 0$ and solve for (x,y). To facilitate the *implicit differentiation* involved, we put $u = f(x,y)$ and use the chain rule for differentiation to obtain

$$\bullet\ \partial_u \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_x f(x,y) + \partial_x \Phi[u,x,y]\big|_{u=f(x,y)} = 0$$
$$\bullet\ \partial_u \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_y f(x,y) + \partial_y \Phi[u,x,y]\big|_{u=f(x,y)} = 0$$

With $\bullet\ \partial_x f(x,y)\big|_{x=x_0,y=y_0} = 0$ and $\bullet\ \partial_y f(x,y)\big|_{x=x_0,y=y_0} = 0$, x_0, y_0 follows from the system of equations

$$\bullet\ \Phi[u,x,y]\big|_{u=f(x,y)} = 0 \text{ for } (x,y) = (x_0,y_0)$$
$$\bullet\ \partial_x \Phi[u,x]\big|_{u=f(x,y)} = 0 \text{ for } (x,y) = (x_0,y_0)$$
$$\bullet\ \partial_y \Phi[u,x]\big|_{u=f(x,y)} = 0 \text{ for } (x,y) = (x_0,y_0)$$

To extract information about $\partial_x^2 f(x,y)\big|_{x=x_0,y=y_0}$, $\partial_y^2 f(x,y)\big|_{x=x_0,y=y_0}$ and $\partial_x\partial_y f(x,y)\big|_{x=x_0,y=y_0}$, the once differentiated expressions are differentiated once more with respect to x and y to yield

- $\partial_u^2 \Phi[u,x,y]\big|_{u=f(x,y)} \cdot [\partial_x f(x,y)]^2 + \partial_u \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_x^2 f(x,y)] +$
  $\partial_x^2 \Phi[u,x,y]\big|_{u=f(x,y)} = 0$
- $\partial_u^2 \Phi[u,x,y]\big|_{u=f(x,y)} \cdot [\partial_y f(x,y)]^2 + \partial_u \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_y^2 f(x,y)] +$
  $\partial_y^2 \Phi[u,x,y]\big|_{u=f(x,y)} = 0$
- $\partial_u^2 \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_x f(x,y) \cdot \partial_y f(x,y) + \partial_u \Phi[u,x,y]\big|_{u=f(x,y)} \cdot \partial_x\partial_y f(x,y)] +$
  $\partial_x\partial_y \Phi[u,x,y]\big|_{u=f(x,y)} = 0$

Continued in Section 07.05b

07.05b INTERIOR EXTREMA OF AN IMPLICIT TARGET FUNCTION OF TWO VARIABLES (CONT'D)

with the result that

- $$\partial_x^2 f(x,y)]\big|_{x=x_0,y-y_0} = -\frac{\partial_x^2 \Phi[u,x,y]\big|_{u=f(x,y),x=x_0,y=y_0}}{\partial_u \Phi[u,x,y]\big|_{u=f(x_0),x=x_0,y=y_0}}$$
- $$\partial_y^2 f(x,y)]\big|_{x=x_0,y-y_0} = -\frac{\partial_y^2 \Phi[u,x,y]\big|_{u=f(x,y),x=x_0,y=y_0}}{\partial_u \Phi[u,x,y]\big|_{u=f(x_0),x=x_0,y=y_0}}$$
- $$\partial_x \partial_y f(x,y)]\big|_{x=x_0,y-y_0} = -\frac{\partial_x \partial_y \Phi[u,x,y]\big|_{u=f(x,y),x=x_0,y=y_0}}{\partial_u \Phi[u,x,y]\big|_{u=f(x_0),x=x_0,y=y_0}}$$

For the rest as to (x_0, y_0) being a minimum or a maximum, the same conditions as in Section 07.04 apply.

$\Longrightarrow$ 08 MATRICES $\Longleftarrow$

08.01 MATRICES

- $M \times N$ *matrix* $[a]$: rectangular (or square) *array* of
  - *elements* $\{a_{m,n}; m = 1, \ldots, M; n = 1, \ldots N\}$:
    - $\begin{bmatrix} a_{1,1} & \ldots & a_{1,N} \\ \ldots & \ldots & \ldots \\ a_{M,1} & \ldots & a_{M,N} \end{bmatrix} = [a_{\text{row index,column index}}] = [a_{m,n}] = [a]$
    - $1 \times N$ matrix $=$ *row* matrix (row vector)
    - $M \times 1$ matrix $=$ *column* matrix (column vector)
    - 1×1 matrix $=$ *scalar*
- *Linear combination* of two $M \times N$ matrices:
  - $[c] = \lambda \cdot [a] + \mu \cdot [b] \Longrightarrow c_{m,n} = \lambda \cdot a_{m,n} + \mu \cdot b_{m,n}$
- *Product matrix* $[c]$ of an $M \times P$ matrix [a] and an $P \times N$ matrix $[b]$:
  - $[c] = [a] * [b] \Longrightarrow c_{m,n} = \sum_{p=1}^{P} a_{m,p} \cdot b_{p,n}$ for $m = 1, \ldots, M; n = 1, \ldots N$
- *Identity matrix (Unit matrix)* $[I]$:
  - $[a] = [I]$ if $a_{m,n} = 1$ for $m = n$ and $a_{m,n} = 0$ for $m \neq n$
- *Zero matrix* $[0]$:
  - $[a] = [0]$ if $a_{m,n} = 0$ for all (m, n)

$\Longrightarrow$ 09 ECONOMIC EXAMPLES AND APPLICATIONS $\Longleftarrow$

09.01 COMPOUND INTEREST

• *Compound interest:* $\Longrightarrow$ Each time interest is paid, its amount is added to (or compounded with) the principal.
• *Periodic* compound interest needs *specification* through:

- the rate of interest per period
- the number of periods of compounding

$$\Longrightarrow \bullet\ FV = PV \cdot (1 + r_{\text{per}})^n$$

with

$$\begin{aligned} FV &= \text{Future Value} \\ PV &= \text{Present Value} \\ r_{\text{per}} &= \text{periodic interest rate} \\ n &= \text{number of periods of compounding } (n = 0, 1, 2, \ldots) \end{aligned}$$

Equivalent formulas:

$$\bullet\ PV = \frac{FV}{(1 + r_{\text{per}})^n}$$

$$\bullet\ r_{\text{per}} = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

$$\bullet\ n = \frac{\ln(FV) - \ln(PV)}{\ln(1 + r_{\text{per}})}$$

• *Continuous* compound interest needs *specification* through:

- the *force of interest*

$$\Longrightarrow \bullet\ A(t) = A(0) \cdot \exp\left[\int_{\tau=0}^{t} \delta(\tau)\mathrm{d}\tau\right]$$

with

$$\begin{aligned} A(t) &= \text{Amount at time } t \\ A(0) &= \text{Amount at } t = 0 \\ \delta(t) &= \frac{\partial_t A(t)}{A(t)} = \text{Force of interest at time } t \end{aligned}$$

Equivalent formula:

$$\bullet\ \delta(t) = \partial_t \ln[A(t)]$$

09.02 EQUIVALENT RATES OF COMPOUND INTEREST

• Equivalence of compound interest rates for accumulated amount over a fixed period of time T.

• *Periodic* compound interest:

$$\Longrightarrow \bullet\ A_{\text{per}}(T) = A(0) \cdot \left(1 + \frac{r_{\text{per}}}{n}\right)^n$$

with

$$\begin{aligned} T &= \text{term of of comparison} \\ A_{\text{per}}(T) &= \text{amount at time } T \\ A(0) &= \text{initial amount} \\ r_{\text{per}} &= \text{periodic interest rate} \\ n &= \text{number of periods in term } T\ (n = 0, 1, 2, \ldots) \end{aligned}$$

• *Continuous* compound interest:

$$\Longrightarrow \bullet\ A_{\delta}(T) = A(0) \cdot \exp(\delta \cdot T)$$

with

$$\begin{aligned} T &= \text{term of of comparison} \\ A_{\delta}(T) &= \text{amount at time } t \\ A(0) &= \text{initial amount} \\ \delta &= \text{(constant) force of interest} \end{aligned}$$

• *Equivalence:*

$$\Longrightarrow \bullet\ A_{\delta}(T) = A_{\text{per}}(T)$$

if

$$\bullet\ \delta = \frac{n}{T} \ln\left(1 + \frac{r_{\text{per}}}{n}\right)$$

or

$$\bullet\ r_{\text{per}} = n \cdot \left[\exp\left(\frac{\delta \cdot T}{n}\right) - 1\right]$$

Exercise. The periodic compound interest rate r_1 payed in n_1 periods during the term T and the periodic compound interest rate r_2 payed in n_2 periods during the same term T are equivalent if

$$\bullet\ \left(1 + \frac{r_1}{n_1}\right)^{n_1} = \left(1 + \frac{r_2}{n_2}\right)^{n_2}$$

Show that (conversion formula)

$$\bullet\ r_2 = n_2 \cdot \left[\left(1 + \frac{r_1}{n_1}\right)^{n_1/n_2}\right] - 1$$

09.03 TIME VALUE OF MONEY, ANNUITY

- *Present value:* money value at a specified time of a sequence of future payments
- *Future value:* value at a specified time in the future assuming a given sequence of payments
- *Annuity:* any recurring periodic sequence of payments.

TIME VALUE OF MONEY (NOMENCLATURE)	
symbol	name
PV	present value
PVA	present value of an annuity
PVP	present value of a perpetuity (perpetual annuity)
FV	future value
FVA	future value of an annuity
FVP	future value of a perpetuity (perpetual annuity)

With

$$\begin{aligned} A &= \text{amount of annuity payment} \\ r_{\text{per}} &= \text{interest rate per period of payment} \\ N &= \text{number of payments} \end{aligned}$$

we have

- $$\begin{aligned} PVA &= A \cdot \sum_{n=1}^{N} \frac{1}{(1+r_{\text{per}})^n} = A \cdot \left[\sum_{n=0}^{N} \frac{1}{(1+r_{\text{per}})^n} - 1 \right] \\ &= A \cdot \frac{1}{r_{\text{per}}} \cdot \left[1 - \frac{1}{(1+r_{\text{per}})^N} \right] \end{aligned}$$
- $$\begin{aligned} FVA &= A \cdot \sum_{n=0}^{N-1} (1+r_{\text{per}})^n \\ &= A \cdot \frac{1}{r_{\text{per}}} \cdot \left[(1+r_{\text{per}})^N - 1 \right] \end{aligned}$$
- $$PVP = \lim_{N\to\infty} PVA = A \cdot \frac{1}{r_{\text{per}}}$$
- $$FVP = \lim_{N\to\infty} FVA = \infty$$

09.04 COBB-DOUGLAS PRODUCTION COST FUNCTION

• *Cobb-Douglas production cost function* for a defined production process:

$$\bullet\; C(N) = \prod_{p=1}^{P} X_p^{\gamma_p/\Gamma}(N) \text{ with } \Gamma = \sum_{p=1}^{P} \gamma_p$$

$$N = \text{number of items produced in the process}$$

$$C(N) = \text{total production costs}$$

$$\{X_p(N); p = 1, \ldots, P\} = \text{sequence of production factors}$$

$$(X_p(N) > 0 \text{ for } N \geq 0,\ p = 1, \ldots, P)$$

$$\gamma_p/\Gamma = \text{relative weighting exponent of } X_p(N) \text{ on } C(N)$$

$$(0 \leq \gamma_p/\Gamma \leq 1)$$

Note: Evidently, $C(N) > 0$ for $N \geq 0$

Note: • $\ln[C(N)] = \sum_{p=1}^{P}(\gamma_p/\Gamma) \cdot \ln[X_p(N)]$

• *Minimum costs per produced unit* $\Longrightarrow$ *minimization of* $C(N)/N$ $\Longrightarrow$

• $\partial_N[C(N)/N] = 0$ and • $\partial_N^2[C(N)/N] > 0$ at $\partial_N[C(N)/N] = 0$

$$\bullet\; \partial_N C(N) = F_{\{X_p\}}(N) \cdot C(N) \text{ with } \bullet\; F_{\{X_p\}}(N) = \sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} \cdot \frac{\partial_N X_p(N)}{X_p(N)}$$

$$\bullet\; \partial_N^2 C(N) = \partial_N F_{\{X_p\}}(N) \cdot C(N) + F_{\{X_p\}}(N) \cdot \partial_N C(N)$$

$$\text{with } \bullet\; \partial_N F_{\{X_p\}}(N) = \sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} \cdot \left[\frac{\partial_N^2 X_p(N)}{X_p(N)} - \frac{[\partial_N X_p(N)]^2}{X_p(N)^2}\right]$$

$$\bullet\; \partial_N\left[\frac{C(N)}{N}\right] = \frac{\partial_N C(N)}{N} - \frac{C(N)}{N^2} = \frac{1}{N} \cdot \left[\partial_N C(N) - \frac{C(N)}{N}\right]$$

$$\bullet\; \partial_N^2\left[\frac{C(N)}{N}\right] = \frac{\partial_N^2 C(N)}{N} - 2 \cdot \frac{\partial_N C(N)}{N^2} + 2 \cdot \frac{C(N)}{N^3}$$

$$= \frac{\partial_N^2 C(N)}{N} - \frac{2}{N^2} \cdot \left[\partial_N C(N) - \frac{C(N)}{N}\right]$$

• *Stationary points:* $\bullet\; F_{\{X_p\}}(N) = \sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} \cdot \frac{\partial_N X_p(N)}{X_p(N)} = \frac{1}{N}$

Linear factors model: • $X_p(N) = a_p + b_p \cdot N$ $(a_p > 0, b_p > 0, p = 1, \ldots, P)$ $\Longrightarrow$

$$\bullet\partial_N^2\left[\frac{C(N)}{N}\right]\Bigg|_{\partial_N[C(N)/N]=0} = \left[-\sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} \cdot \left(\frac{b_p}{a_p + b_p \cdot N}\right)^2 + \frac{1}{N^2}\right] \cdot \frac{C(N)}{N} > 0$$

$$\text{since } \bullet\; \sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} \cdot \left(\frac{b_p}{a_p + b_p \cdot N}\right)^2 < \frac{1}{N^2} \cdot \sum_{p=1}^{P} \frac{\gamma_p}{\Gamma} = \frac{1}{N^2}$$

$\Longrightarrow$ The (single) stationary point of $C(N)/N$ is a minimum

⟹ 10 MISCELLANEOUS EXERCISES ⟸

10.01 EXERCISE

Let

$$\bullet f(x) = \left(\frac{x}{a}\right)^p - 1 \text{ with } a > 0,\ p = \text{integer},\ \text{for } 0 \le x < \infty.$$

Investigate in which of the different domains of the parameters a and p and the variable x we have (a) $f(x) < 0$, (b) $f(x) = 0$, (c) $f(x) > 0$. (See, the table below).

	$f(x) = (x/a)^p - 1$			
	$x = 0$	$0 < x < a$	$x = a$	$a < x < \infty$
$p < 0$				
$p = 0$				
$p > 0$				

10.02 EXERCISE

Let

$$\bullet f(x) = \left|\left(\frac{x}{a}\right)^p - 1\right| \text{ with } a > 0,\ p = \text{integer},\ \text{for } 0 \le x < \infty.$$

Write the expressions for $f(x)$ that hold in the different domains of the parameters a and p and the variable x. (See, the table below).

	$f(x) = \lvert(x/a)^p - 1\rvert$			
	$x = 0$	$0 < x < a$	$x = a$	$a < x < \infty$
$p < 0$				
$p = 0$				
$p > 0$				

(*Hint:* Use Sections 02.01 and 02.03.)

10.03 EXERCISE

How does one compute $f(x) = x^p$ with $x > 0$ for any (also non-integer) value of p?
(*Hint:* Use the natural logarithm, see Section 06.05.)

10.04 EXERCISE

Show that Newton's binomial coefficient can be written as

$$\bullet B_n^N = \frac{N!}{n! \cdot (N-n)!} \text{ for } n = 0, 1, \ldots, N.$$

(*Hint:* Verify that $B_0^{N+1} = B_0^N = 1$, $B_{N+1}^{N+1} = B_N^N = 1$ and $B_n^{N+1} = B_{n-1}^N + B_n^N$ for $n = 1, \ldots, N$, see Sections 02.04 and 05.01.)

10.05 EXERCISE

Given the sequence

$$\bullet \ \{0, 1, 2, \ldots, 98, 99, 100\}.$$

Evaluate the sum of the (a) *odd-valued*, (b)*even-valued*, (c) *all* elements of the sequence. (*Hint:* Use the expression for the sum of an arithmetic sequence, see Section 02.07.)

10.06 EXERCISE

Evaluate

$$\bullet \ \sum_{n=0}^{10} 2^n \qquad \bullet \ \sum_{n=0}^{10} (-2)^{-n}$$

$$\bullet \ \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \qquad \bullet \ \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$$

(*Hint:* Use the expression for the sum of a geometric sequence, see Section 02.09.)

10.07 EXERCISE

Derive the Taylor expansions of the following functions

$$\bullet \ f(x) = x^2 + x + 1$$

$$\bullet \ g(x) = 2 \cdot x^2 + 4 \cdot x + 2$$

about (a) $x = -1$, (b) $x = 1$, and check the result.
(*Hint:* Use the expression for the Taylor expansion, see Section 05.06.)

10.08 EXERCISE

Derive the Taylor expansion of the function

- $f(x) = a \cdot x^2 + b \cdot x + c$

about (a) $x = \xi$ and check the result.
(*Hint:* Use the expression for the Taylor expansion, see Section 05.06.)

10.09 EXERCISE

Show that for any x

- $$(b-a)^N = \sum_{n=0}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot (x-a)^{N-n} \cdot (b-x)^n.$$

(*Hint:* Use Newton's binomial theorem, Section 05.01, and the result of Exercise 10.04.)
Note: This result can be used to establish a number of relationships between expressions containing factorial functions.

10.10 EXERCISE

Let

- $$f(x) = \sum_{n=0}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot (x-a)^{N-n} \cdot (b-x)^n.$$

Show that $\partial_x f(x) = 0$. (*Hint:* Apply the product rule for differentiation to each of the terms in the summation and use Newton's binomial theorem (Section 05.01) to rewrite the result. See also Exercise 10.09.)

10.11 EXERCISE

Check the product rule for differentiation for the product $f(x) \cdot g(x)$, with

- $f(x) = a \cdot x + b$
- $g(x) = c \cdot x + d$

10.12 EXERCISE

Check the chain rule for differentiation for

- $f(x) = (a \cdot x + b)^N \quad N = 0, 1, 2, \ldots$

against differentiating each term of Newton's binomial expansion of the right-hand side.

10.13 EXERCISE

Show that (x_0, y_0) is a stationary point of

- $f(x, y) = A \cdot (x - x_0)^2 + 2 \cdot B \cdot (x - x_0) \cdot (y - y_0) + C \cdot (y - y_0)^2 + F$

Under what conditions to be laid upon A, B and C is (x_0, y_0) (a) a minimum, (b) a maximum?

10.13 EXERCISE

Determine the stationary point (x_0, y_0) of

- $f(x, y) = A \cdot x^2 + 2 \cdot B \cdot x \cdot y + C \cdot y^2 + D \cdot x + E \cdot y + F$

Under what conditions to be laid upon A, B and C is (x_0, y_0) (a) a minimum, (b) a maximum?

10.15 EXERCISE

Evaluate

- $I(s, n) = \int_{t=0}^{\infty} \exp(-s \cdot t) \cdot t^n \mathrm{d}t$ with $s > 0$ and $n = 0, 1, 2, \ldots$

by integration by parts and recursion.

10.15 EXERCISE

Evaluate

- $I(s,n) = \int_{t=0}^{\infty} \exp(-s \cdot t) \cdot t^n \mathrm{d}t$ with $s > 0$ and $n = 0, 1, 2, \ldots$

by differentiating

- $I(s,0) = \int_{t=0}^{\infty} \exp(-s \cdot t) \mathrm{d}t = \frac{1}{s}$ with $s > 0$

n times with respect to the parameter s.

10.17 EXERCISE

Evaluate

- $I = \int_{x=\ln(a)}^{\ln(b)} \exp(x) \mathrm{d}x$ with $a > 0, b > 0$

10.18 EXERCISE

Evaluate

- $I = \int_{x=\ln(a)}^{\ln(b)} \exp(-x) \mathrm{d}x$ with $a > 0, b > 0$

10.19 EXERCISE

Evaluate

- $I = \int_{x=\ln(a)}^{\ln(b)} \exp(x^2) \cdot x \mathrm{d}x$ with $a > 0, b > 0$

10.20 EXERCISE

Evaluate

- $I = \int_{x=\ln(a)}^{\ln(b)} \exp(-x^2) \cdot x \mathrm{d}x$ with $a > 0, b > 0$

10.21 EXERCISE

Evaluate

- $I = \int_{x=a}^{b} \ln(x+p)\mathrm{d}x$ with $a > 0, b > 0, p > 0$

10.22 EXERCISE

Evaluate

- $I = \int_{x=a}^{b} \ln(x^2+p^2)\cdot x\mathrm{d}x$ with $a > 0, b > 0, p > 0$

10.23 EXERCISE

Evaluate

- $I = \int_{x=a}^{b} \frac{x}{(x^2+p^2)^{1/2}}\mathrm{d}x$ with $a > 0, b > 0, p > 0$

10.24 EXERCISE

Evaluate

- $I = \int_{x=a}^{b} \frac{1}{(x^2+p^2)^{1/2}}\mathrm{d}x$ with $a > 0, b > 0, p > 0$

(*Hint:* Observe (and verify) that $\partial_x \ln[x+(x^2+p^2)^{1/2}] = (x^2+p^2)^{-1/2}$.)

10.25 EXERCISE

For a closed economy, the following conservation equation holds:

- $Y(I) = I + C(Y)$

where

- Y = national income
- I = investment
- C = consumption

The consumption function has the properties

- $C(Y) \geq 0$ for $Y \geq 0$
- $C(Y) \leq Y$ for $Y \geq 0$
- $\lim_{Y \to \infty} C(Y) = C_\infty < \infty$
- $\partial_Y C(Y) \leq 1$ for $Y \geq 0$
- $\partial_Y^2 C(Y) < 0$ for $Y \geq 0$

(a) Make a graph with Y along the horizontal axis, C along the vertical axis and I as a parameter. (b) Determine an expression for $\partial_I Y$. (c) Determine an expression for $\partial_I^2 Y$. (d) Discuss the results.

10.26 EXERCISE

A profit function of a production process is characterized by

- $\pi(N) = P \cdot N - C(N)$

where

- π = profit
- P = sales price per item
- N = number of produced items
- C = production cost function

The production cost function per item has the properties

- $C(N) \geq 0$ for $N \geq 0$
- $\partial_N C(N) > 0$ for $N \geq 0$
- $\partial_N^2 C(N) > 0$ for $N \geq 0$

(a) Make a graph with N along the horizontal axis and C along the vertical axis. (b) Determine an expression for $\partial_N \pi$. (c) Determine an expression for $\partial_N^2 \pi$. (d) Give the equations for the optimum value N^* of N. (e) Is $\pi(N^*)$ a maximum of $\pi(N)$? (f) Discuss the results.

10.27 EXERCISE

A bond is issued on the following conditions:

- A_P = principal to be returned to owner after N periods of payment
- N = number of periods of payment
- P = amount of payment per period
- r_{eff} = effective interest rate per period

(a) Determine the present value PV of the bond at the time of issue.
(b) The bond issuing agency takes into consideration the following possibilities of changing the conditions: (i) replacing P with $P_1 > P$ and keeping A the same; (ii) replacing A_P with $A_{P,2} > A$ and keeping P the same. Determine for both cases the corresponding present values PV_1 for (i) and PV_2 for (ii).
(c) What is the relation between P_1 and $A_{P,2}$ for $PV_1 = PV_2$?

$\Longrightarrow$ 10 MISCELLANEOUS EXERCISES (ANSWERS) $\Longleftarrow$

10.01 EXERCISE (ANSWER)

Answer:
(a), (b) and (c), see table below:

	$f(x) = (x/a)^p - 1$			
	$x = 0$	$0 < x < a$	$x = a$	$a < x < \infty$
$p < 0$	$f(x) < 0$	$f(x) > 0$	$f(x) = 0$	$f(x) < 0$
$p = 0$	$f(x) = 0$	$f(x) = 0$	$f(x) = 0$	$f(x) = 0$
$p > 0$	$f(x) < 0$	$f(x) < 0$	$f(x) = 0$	$f(x) > 0$

Note: $0^0 = 1$.

10.02 EXERCISE (ANSWER)

Answer:
See table below:

	$f(x) = \|(x/a)^p - 1\|$			
	$x = 0$	$0 < x < a$	$x = a$	$a < x < \infty$
$p < 0$	$f(x) = -1$	$f(x) = (x/a)^p - 1$	$f(x) = 0$	$f(x) = 1 - (x/a)^p$
$p = 0$	$f(x) = 0$	$f(x) = 0$	$f(x) = 0$	$f(x) = 0$
$p > 0$	$f(x) = -1$	$f(x) = 1 - (x/a)^p$	$f(x) = 0$	$f(x) = (x/a)^p - 1$

Note: $0^0 = 1$.

10.03 EXERCISE (ANSWER)

Answer:

$$f(x) = \exp[p\ln(x)] \quad (x > 0)$$

10.04 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
B_0^{N+1} &= \frac{(N+1)!}{0!\cdot(N+1)!} = \frac{(N+1)!}{1\cdot(N+1)!} = 1,\\
B_0^{N} &= \frac{N!}{0!\cdot N!} = \frac{N!}{1\cdot N!} = 1,\\
B_{N+1}^{N+1} &= \frac{(N+1)!}{(N+1)!\cdot 0!} = \frac{(N+1)!}{(N+1)!\cdot 1} = 1,\\
B_N^{N} &= \frac{N!}{N!\cdot 0!} = \frac{N!}{N!\cdot 1} = 1,\\
B_{n-1}^{N} + B_n^{N} = &= \frac{N!}{(n-1)!\cdot(N-n+1)!} + \frac{N!}{n!\cdot(N-n)!}\\
&= \frac{(N+1)!}{n!\cdot(N-n+1)!}\cdot\left(\frac{n}{N+1} + \frac{N-n+1}{N+1}\right)\\
&= \frac{(N+1)!}{n!\cdot(N-n+1)!}\cdot 1\\
&= B_n^{N+1}
\end{aligned}
$$

10.05 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
S^{\text{odd}} &= \frac{1}{2}\cdot 50\cdot(1+99) = 2500\\
S^{\text{even}} &= \frac{1}{2}\cdot 51\cdot(0+100) = 2550 \text{ or, upon omitting } 0,\\
S^{\text{even}} &= \frac{1}{2}\cdot 50\cdot(2+100) = 2550\\
S^{\text{all}} &= \frac{1}{2}\cdot 101\cdot(0+100) = 5050 \text{ or, upon omitting } 0,\\
S^{\text{all}} &= \frac{1}{2}\cdot 100\cdot(1+100) = 5050
\end{aligned}
$$

Check: $S^{\text{odd}} + S^{\text{even}} = S^{\text{all}}$

10.06 EXERCISE (ANSWER)

Answer:

$$\sum_{n=0}^{10} 2^n = 1 \cdot \frac{1-2^{10+1}}{1-2} = \frac{1-2^{11}}{1-2} = \frac{1-2048}{1-2} = \frac{-2047}{-1} = 2047$$

$$\begin{aligned}\sum_{n=0}^{10}(-2)^{-n} &= \sum_{n=0}^{1}0\left(-\frac{1}{2}\right)^n \\ &= 1 \cdot \frac{1-(-1/2)^{10+1}}{1-(-1/2)} = \frac{1-(-1/2)^{11}}{1+1/2} = \frac{1+1/2048}{1+1/2} \\ &= \frac{2048+1}{2048} \cdot \frac{2}{3} = \frac{683}{1024}\end{aligned}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 \cdot \frac{1}{1-(1/2)} = 2$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 \cdot \frac{1}{1-(-1/2)} = \frac{2}{3}$$

10.07 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} f(x) &= x^2 + x + 1, \\ \partial_x f(x) &= 2 \cdot x + 1, \quad \partial_x^2 f(x) = 2, \quad \partial_x^n f(x) = 0 \text{ for } n = 3, 4, 5, \ldots \\ g(x) &= 2 \cdot x^2 + 4 \cdot x + 2, \\ \partial_x g(x) &= 2 \cdot 2 \cdot x + 4 = 4 \cdot x + 4, \quad \partial_x^2 g(x) = 4, \quad \partial_x^n g(x) = 0 \text{ for } n = 3, 4, 5, \ldots \end{aligned}$$

(a) Taylor expansion about $x = -1$:

$$\begin{aligned} f(x) &= f(x)\big|_{x=-1} + \frac{x+1}{1!} \cdot \partial_x f(x)\big|_{x=-1} + \frac{(x+1)^2}{2!} \cdot \partial_x^2 f(x)\big|_{x=-1} \\ &= 1 - (x+1) + 2 \cdot \frac{(x+1)^2}{2} = 1 - (x+1) + (x+1)^2 \\ &= x^2 + x + 1 = f(x) \text{ as given} \\ g(x) &= g(x)\big|_{x=-1} + \frac{x+1}{1!} \cdot \partial_x g(x)\big|_{x=-1} + \frac{(x+1)^2}{2!} \cdot \partial_x^2 g(x)\big|_{x=-1} \\ &= 0 - 0 \cdot (x+1) + 4 \cdot \frac{(x+1)^2}{2} = 2 \cdot (x+1)^2 \\ &= 2 \cdot x^2 + 4 \cdot x + 2 = g(x) \text{ as given} \end{aligned}$$

(b) Taylor expansion about $x = 1$:

$$\begin{aligned} f(x) &= f(x)\big|_{x=1} + \frac{x-1}{1!} \cdot \partial_x f(x)\big|_{x=1} + \frac{(x-1)^2}{2!} \cdot \partial_x^2 f(x)\big|_{x=1} \\ &= 3 + 3 \cdot (x-1) + 2 \cdot \frac{(x-1)^2}{2} = 3 + 3 \cdot (x-1) + (x-1)^2 \\ &= x^2 + x + 1 = f(x) \text{ as given} \\ g(x) &= g(x)\big|_{x=1} + \frac{x-1}{1!} \cdot \partial_x g(x)\big|_{x=1} + \frac{(x-1)^2}{2!} \cdot \partial_x^2 g(x)\big|_{x=1} \\ &= 8 + 8 \cdot (x-1) + 4 \cdot \frac{(x-1)^2}{2} = 8 + 8 \cdot (x-1) + 2 \cdot (x-1)^2 = \\ &= 2 \cdot x^2 + 4 \cdot x + 2 = g(x) \text{ as given} \end{aligned}$$

10.08 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} f(x) &= a \cdot x^2 + b \cdot x + c, \\ \partial_x f(x) &= 2 \cdot a \cdot x + b, \quad \partial_x^2 f(x) = 2 \cdot a, \quad \partial_x^n f(x) = 0 \text{ for } n = 3, 4, 5, \ldots \end{aligned}$$

Taylor expansion about $x = \xi$:

$$\begin{aligned} f(x) &= f(x)\big|_{x=\xi} + \frac{x-\xi}{1!} \cdot \partial_x f(x)\big|_{x=\xi} + \frac{(x-\xi)^2}{2!} \cdot \partial_x^2 f(x)\big|_{x=\xi} \\ &= a \cdot \xi^2 + b \cdot \xi + c + (2 \cdot a \cdot \xi + b) \cdot (x - \xi) + 2 \cdot a \cdot \frac{(x-\xi)^2}{2!} \\ &= a \cdot \xi^2 + b \cdot \xi + c + 2 \cdot a \cdot \xi \cdot x - 2 \cdot a \cdot \xi^2 + b \cdot x - b \cdot \xi + \\ &\quad\; a \cdot x^2 - 2 \cdot a \cdot x \cdot \xi + a \cdot \xi^2 \\ &= a \cdot x^2 + b \cdot x + c = f(x) \text{ as given} \end{aligned}$$

10.09 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} (b-a)^N &= [(x-a) + (b-x)]^N \\ &= \sum_{n=0}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot (x-a)^{N-n} \cdot (b-x)^n. \end{aligned}$$

10.10 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} \partial_x f(x) &= \sum_{n=0}^{N-1} \frac{N!}{n! \cdot (N-n)!} \cdot (N-n) \cdot (x-a)^{N-n-1} \cdot (b-x)^n - \\ &\quad \sum_{n=1}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot (x-a)^{N-n} \cdot n \cdot (b-x)^{n-1} \\ &= \sum_{n=0}^{N-1} \frac{N!}{n! \cdot (N-n-1)!} \cdot (x-a)^{N-n-1} \cdot (b-x)^n - \\ &\quad \sum_{m=0}^{N-1} \frac{N!}{m! \cdot (N-m-1)!} \cdot (x-a)^{N-m-1} \cdot (b-x)^m \\ &= 0 \end{aligned}$$

10.11 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
f(x) &= a \cdot x + b \quad \Longrightarrow \partial_x f(x) = a \\
g(x) &= c \cdot x + d \quad \Longrightarrow \partial_x g(x) = c \\
f(x) \cdot g(x) &= (a \cdot x + b) \cdot (c \cdot x + d) \\
&= a \cdot c \cdot x^2 + (b \cdot c + a \cdot d) \cdot x + b \cdot d \\
\partial_x[f(x) \cdot g(x)] &= 2 \cdot a \cdot c \cdot x + b \cdot c + a \cdot d \ (*)
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\partial_x f(x) \cdot g(x) + f(x) \cdot \partial_x f(x) &= a \cdot (c \cdot x + d) + (a \cdot x + b) \cdot c \\
&= 2 \cdot a \cdot c \cdot x + a \cdot d + b \cdot c \ (**) \\
(*) &= (**) \ \text{(Product rule for differentiation)}
\end{aligned}
$$

10.12 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
f(x) &= (a \cdot x + b)^N \quad \Longrightarrow \\
\partial_x f(x) &= N \cdot (a \cdot x + b)^{N-1} \cdot \partial_x (a \cdot x + b) \\
&= N \cdot (a \cdot x + b)^{N-1} \cdot a \ (*)
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot (a \cdot x)^{N-n} \cdot b^n \Longrightarrow \\
\partial_x f(x) &= \sum_{n=0}^{N} \frac{N!}{n! \cdot (N-n)!} \cdot \partial_x (a \cdot x)^{N-n} \cdot b^n \\
&= \sum_{n=0}^{N-1} \frac{N!}{n! \cdot (N-n)!} \cdot (N-n) \cdot (a \cdot x)^{N-n-1} \cdot \partial_x (a \cdot x) \cdot b^n \\
&= \sum_{n=0}^{N-1} \frac{N!}{n! \cdot (N-n)!} \cdot (N-n) \cdot a^{N-n} \cdot x^{N-n-1} \cdot b^n \\
&= \sum_{n=0}^{N-1} \frac{N!}{n! \cdot (N-n-1)!} \cdot a^{N-n} \cdot x^{N-n-1} \cdot b^n \\
&= \sum_{n=0}^{N-1} N \cdot \frac{(N-1)!}{n! \cdot (N-n-1)!} \cdot a^{N-n} \cdot x^{N-n-1} \cdot b^n \\
&= \sum_{n=0}^{N-1} N \cdot a \cdot \frac{(N-1)!}{n! \cdot (N-n-1)!} \cdot a^{N-n-1} \cdot x^{N-n-1} \cdot b^n \\
&= N \cdot a \cdot \sum_{n=0}^{N-1} \frac{(N-1)!}{n! \cdot (N-n-1)!} \cdot (a \cdot x)^{N-n-1} \cdot b^n \\
&= N \cdot a \cdot (a \cdot x + b)^{N-1} \quad (**) \\
(*) &= (**) \text{ Chain rule for differentiation)}
\end{aligned}
$$

10.13 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} f(x,y) &= A\cdot(x-x_0)^2 + 2\cdot B\cdot(x-x_0)\cdot(y-y_0) + C\cdot(y-y_0)^2 + F \Longrightarrow \\ \partial_x f(x,y) &= 2\cdot A\cdot(x-x_0) + 2\cdot B\cdot(y-y_0), \quad \partial_x^2 f(x,y) = 2\cdot A, \quad \partial_y\partial_x f(x,y) = 2\cdot B \\ \partial_y f(x,y) &= 2\cdot B\cdot(x-x_0) + 2\cdot C\cdot(y-y_0), \quad \partial_y^2 f(x,y) = 2\cdot C, \quad \partial_x\partial_y f(x,y) = 2\cdot B \\ &\Longrightarrow \partial_x f(x,y)\big|_{x=x_0,y=y_0} = 0, \quad \partial_y f(x,y)\big|_{x=x_0,y=y_0} = 0 \\ &\Longrightarrow (x_0,y_0) = \text{stationary point} \end{aligned}$$

(a) Minimum if

$$A>0, \quad C>0, \quad A\cdot C - B^2 > 0$$

(b) Maximum if

$$A<0, \quad C<0, \quad A\cdot C - B^2 > 0$$

10.14 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} f(x,y) &= A\cdot x^2 + 2\cdot B\cdot x\cdot y + C\cdot y^2 + D\cdot x + e\cdot y + F \Longrightarrow \\ \partial_x f(x,y) &= 2\cdot A\cdot x + 2\cdot B\cdot y + D, \quad \partial_x^2 f(x,y) = 2\cdot A, \quad \partial_y\partial_x f(x,y) = 2\cdot B \\ \partial_y f(x,y) &= 2\cdot B\cdot x + 2\cdot C\cdot y + E, \quad \partial_y^2 f(x,y) = 2\cdot C, \quad \partial_x\partial_y f(x,y) = 2\cdot B \end{aligned}$$

Stationary point:

$$\begin{aligned} \partial_x f(x,y)\big|_{x=x_0,y=y_0} &= 0 \Longrightarrow 2\cdot A\cdot x_0 + 2\cdot B\cdot y_0 + D = 0 \\ \partial_y f(x,y)\big|_{x=x_0,y=y_0} &= 0 \Longrightarrow 2\cdot B\cdot x_0 + 2\cdot C\cdot y_0 + E = 0 \\ &\Longrightarrow \\ x_0 &= \frac{B\cdot E - C\cdot D}{2\cdot(A\cdot C - B^2)}, \quad y_0 = \frac{B\cdot D - A\cdot E}{2\cdot(A\cdot C - B^2)} \end{aligned}$$

(a) Minimum if

$$A>0, \quad C>0, \quad A\cdot C - B^2 > 0$$

(b) Maximum if

$$A<0, \quad C<0, \quad A\cdot C - B^2 > 0$$

10.15 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I(s,n) &= \int_{t=0}^{\infty} \exp(-s\cdot t)\cdot t^n \mathrm{d}t \\
&= -\frac{1}{s}\cdot \int_{t=0}^{\infty} \partial_t \exp(-s\cdot t)\cdot t^n \mathrm{d}t \\
&= -\frac{1}{s}\cdot \left[\exp(-s\cdot t)\cdot t^n\right]\Big|_{t=0}^{\infty} \mathrm{d}t \\
&\quad +\frac{1}{s}\cdot \int_{t=0}^{\infty} \exp(-s\cdot t)\cdot \partial_t t^n \mathrm{d}t \text{ for } n=1,2,3,\ldots \\
&= \frac{n}{s}\cdot \int_{t=0}^{\infty} \exp(-s\cdot t)\cdot t^{n-1} \mathrm{d}t \text{ for } n=1,2,3,\ldots \\
&= \frac{n}{s}\cdot I(s,n-1) \text{ for } n=1,2,3,\ldots \\
I(s,0) &= \int_{t=0}^{\infty} \exp(-s\cdot t)\mathrm{d}t \\
I(s,0) &= -\frac{1}{s}\cdot \int_{t=0}^{\infty} \partial_t \exp(-s\cdot t)\mathrm{d}t \\
&= -\frac{1}{s}\cdot \left[\exp(-s\cdot t)\right]\Big|_{t=0}^{\infty} \mathrm{d}t \\
&= \frac{1}{s} \\
&\Longrightarrow \\
I(s,n) &= \frac{n!}{s^{n+1}}
\end{aligned}
$$

10.16 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I(s,n) &= \int_{t=0}^{\infty} \exp(-s\cdot t)\cdot t^n \mathrm{d}t \\
I(s,0) &= \int_{t=0}^{\infty} \exp(-s\cdot t)\mathrm{d}t \\
&= -\frac{1}{s}\cdot \int_{t=0}^{\infty} \partial_t \exp(-s\cdot t)\mathrm{d}t \\
&= -\frac{1}{s}\cdot [\exp(-s\cdot t)]\Big|_{t=0}^{\infty} \mathrm{d}t \\
&= \frac{1}{s} \\
\partial_s I(s,0) &= \int_{t=0}^{\infty} \partial_s \exp(-s\cdot t)\mathrm{d}t = \partial_s\left(\frac{1}{s^2}\right) \\
&= -\int_{t=0}^{\infty} \exp(-s\cdot t)\cdot t\mathrm{d}t = -\frac{1}{s^2} \\
&\Longrightarrow \\
I(s,n) &= \frac{n!}{s^{n+1}} \text{ for } n = 0,1,2,\ldots
\end{aligned}
$$

10.17 EXERCISE (ANSWER)

Answer:

- $$
\begin{aligned}
I &= \int_{x=\ln(a)}^{\ln(b)} \exp(x)\mathrm{d}x \\
&= \int_{x=\ln(a)}^{\ln(b)} \partial_x \exp(x)\ \mathrm{d}x \\
&= \exp(x)\Big|_{x=\ln(a)}^{\ln(b)} \\
&= \exp[\ln(b)] - \exp[\ln(a)] \\
&= b - a
\end{aligned}
$$

10.18 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
\bullet\ I &= \int_{x=\ln(a)}^{\ln(b)} \exp(-x)\mathrm{d}x \\
&= -\int_{x=\ln(a)}^{\ln(b)} \partial_x \exp(x)\ \mathrm{d}x \\
&= -\exp(-x)\Big|_{x=\ln(a)}^{\ln(b)} \\
&= -\exp[-\ln(b)] + \exp[-\ln(a)] \\
&= -\frac{1}{b} + \frac{1}{a}
\end{aligned}
$$

10.19 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
\bullet\ I &= \int_{x=\ln(a)}^{\ln(b)} \exp(x^2)\cdot x\ \mathrm{d}x \\
&= \frac{1}{2}\cdot\int_{x=\ln(a)}^{\ln(b)} \exp(x^2)\cdot\partial_x x^2\ \mathrm{d}x \\
&\overset{x^2\to u}{=} \frac{1}{2}\cdot\int_{u=[\ln(a)]^2}^{[\ln(b)]^2} \exp(u)\ \mathrm{d}u \\
&= \frac{1}{2}\cdot\exp(u)\Big|_{u=[\ln(a)]^2}^{[\ln(b)]^2} \\
&= \frac{1}{2}\cdot\left[\exp\{[\ln(b)]^2\} - \exp\{[\ln(a)]^2\}\right]
\end{aligned}
$$

10.20 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I &= \int_{x=\ln(a)}^{\ln(b)} \exp(-x^2) \cdot x \, \mathrm{d}x \\
&= \frac{1}{2} \cdot \int_{x=\ln(a)}^{\ln(b)} \exp(-x^2) \cdot \partial_x x^2 \, \mathrm{d}x \\
&\overset{x^2 \to u}{=} \frac{1}{2} \cdot \int_{u=[\ln(a)]^2}^{[\ln(b)]^2} \exp(-u) \, \mathrm{d}u \\
&= -\frac{1}{2} \cdot \exp(-u) \Big|_{u=[\ln(a)]^2}^{[\ln(b)]^2} \\
&= \frac{1}{2} \cdot \left[-\exp\{-[\ln(b)]^2\} + \exp\{-[\ln(a)]^2\} \right]
\end{aligned}
$$

10.21 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I &= \int_{x=a}^{b} \ln(x+p) \mathrm{d}x \\
&= \int_{x=a}^{b} \ln(x+p) \cdot \partial_x (x+p) \, \mathrm{d}x \\
&= [\ln(x+p) \cdot (x+p)] \Big|_{x=a}^{b} - \int_{x=a}^{b} \partial_x [\ln(x+p)] \cdot (x+p) \, \mathrm{d}x \\
&= [\ln(x+p) \cdot (x+p)] \Big|_{x=a}^{b} - \int_{x=a}^{b} \frac{1}{x+p} \cdot (x+p) \, \mathrm{d}x \\
&= [\ln(x+p) \cdot (x+p)] \Big|_{x=a}^{b} - \int_{x=a}^{b} 1 \, \mathrm{d}x \\
&= [\ln(x+p) \cdot (x+p)] \Big|_{x=a}^{b} - x \Big|_{x=a}^{b} \\
&= \ln(b+p) \cdot (b+p) - \ln(a+p) \cdot (a+p) - (b-a)
\end{aligned}
$$

10.22 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I &= \int_{x=a}^{b} \ln(x^2+p^2)\cdot x\mathrm{d}x \\
&= \frac{1}{2}\cdot\int_{x=a}^{b} \ln(x^2+p^2)\cdot\partial_x(x^2+p^2)\ \mathrm{d}x \\
&\overset{x^2\to u}{=} \frac{1}{2}\cdot\left[\ln(u+p^2)\cdot(u+p^2)\right]\Big|_{u=a^2}^{b^2} - \frac{1}{2}\cdot\int_{u=a^2}^{b^2} \partial_u[\ln(u+p^2)]\cdot(u+p^2)\ \mathrm{d}u \\
&= \left[\ln(u+p^2)\cdot(u+p^2)\right]\Big|_{u=a^2}^{b^2} - \int_{u=a^2}^{b^2} \frac{1}{u+p^p}\cdot(u+p^2)\ \mathrm{d}x \\
&= \frac{1}{2}\cdot\left[\ln(u+p^2)\cdot(u+p^2)\right]\Big|_{u=a^2}^{b^2} - \frac{1}{2}\cdot\int_{u=a^2}^{b^2} 1\ \mathrm{d}u \\
&= \frac{1}{2}\cdot\left[\ln(u+p^2)\cdot(u+p^2)\right]\Big|_{u=a^2}^{b^2} - \frac{1}{2}\cdot u\Big|_{u=a^2}^{b^2} \\
&= \frac{1}{2}\cdot\ln(b^2+p^2)\cdot(b^2+p^2) - \frac{1}{2}\cdot\ln(a^2+p^2)\cdot(a^2+p^2) - \frac{1}{2}\cdot(b^2-a^2)
\end{aligned}
$$

10.23 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I &= \int_{x=a}^{b} \frac{x}{(x^2+p^2)^{1/2}}\mathrm{d}x \\
&= \frac{1}{2}\cdot\int_{x=a}^{b} \partial_x\left[(x^2+p^2)^{1/2}\right]\ \mathrm{d}x \\
&= \frac{1}{2}\cdot\left[(x^2+p^2)^{1/2}\right]\Big|_{x=a}^{b} \\
&= \frac{1}{2}\cdot\left[(b^2+p^2)^{1/2}-(a^2+p^2)^{1/2}\right]
\end{aligned}
$$

10.24 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
I &= \int_{x=a}^{b} \frac{1}{(x^2+p^2)^{1/2}}\,\mathrm{d}x \\
&= \int_{x=a}^{b} \partial_x \ln[x+(x^2+p^2)^{1/2}]\,\mathrm{d}x \\
&= \ln[x+(x^2+p^2)^{1/2}]\Big|_{x=a}^{b} \\
&= \ln[b+(b^2+p^2)^{1/2}] - \ln[a+(a^2+p^2)^{1/2}] \\
&= \ln\left[\frac{b+(b^2+p^2)^{1/2}}{a+(a^2+p^2)^{1/2}}\right]
\end{aligned}
$$

10.25 EXERCISE (ANSWER)

Answer:

$$
\begin{aligned}
Y(I) &= I + C(Y) \\
\partial_I Y(I) &= 1 + \partial_Y C(Y)\cdot\partial_I Y(I) \\
&\Longrightarrow \\
\partial_I Y(I) &= \frac{1}{1+\partial_Y C(Y)}. \\
\partial_I^2 Y(I) &= \partial_Y^2 C(Y)\cdot[\partial_I Y(I)]^2 + \partial_Y C(Y)\cdot\partial_I^2 Y(I) \\
&\Longrightarrow \\
\partial_I^2 Y(I) &= \frac{\partial_Y^2 C(Y)\cdot[\partial_I Y(I)]^2}{1+\partial_Y C(Y)} \\
&= \frac{\partial_Y^2 C(Y}{[1+\partial_Y C(Y)]^3}.
\end{aligned}
$$

10.26 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} \pi(N) &= P \cdot N - C(N) \\ \partial_N \pi(N) &= P - \partial_N C(N) \\ \partial_N^2 \pi(N) &= -\partial_N^2 C(N) \end{aligned}$$

Stationary point $N = N^*$:

$$\begin{aligned} \pi(N^*) &= P \cdot N^* - C(N^*) \\ \partial_N \pi(N)\big|_{N=N^*} &= 0 \\ &\Longrightarrow \\ 0 &= P - \partial_N C(N)\big|_{N=N^*} \end{aligned}$$

Since $\partial_N^2 C(N) > 0$, $N = N^*$ is a maximum 0f $\pi(N)$.

10.27 EXERCISE (ANSWER)

Answer:

$$\begin{aligned} PV &= \frac{A_P}{(1+r_{\text{eff}})^N} + \sum_{n=1}^{N} \frac{P}{(1+r_{\text{eff}})^n} \\ &= \frac{A_P}{(1+r_{\text{eff}})^N} + \frac{P}{1+r_{\text{eff}}} \cdot \sum_{m=0}^{N-1} \frac{1}{(1+r_{\text{eff}})^m} \\ &= \frac{A_P}{(1+r_{\text{eff}})^N} + \frac{P}{1+r_{\text{eff}}} \cdot \frac{1-(1+r_{\text{eff}})^{-N}}{1-(1+r_{\text{eff}})^{-1}} \\ PV_1 &= \frac{A_P}{(1+r_{\text{eff}})^N} + \frac{P_1}{1+r_{\text{eff}}} \cdot \frac{1-(1+r_{\text{eff}})^{-N}}{1-(1+r_{\text{eff}})^{-1}} \\ PV_2 &= \frac{A_{P,2}}{(1+r_{\text{eff}})^N} + \frac{P}{1+r_{\text{eff}}} \cdot \frac{1-(1+r_{\text{eff}})^{-N}}{1-(1+r_{\text{eff}})^{-1}} \\ &\Longrightarrow \end{aligned}$$

$PV_1 = PV_2$ for

$$\frac{A_P}{(1+r_{\text{eff}})^N} + \frac{P_1}{1+r_{\text{eff}}} \cdot \frac{1-(1+r_{\text{eff}})^{-N}}{1-(1+r_{\text{eff}})^{-1}} = \frac{A_{P,2}}{(1+r_{\text{eff}})^N} + \frac{P}{1+r_{\text{eff}}} \cdot \frac{1-(1+r_{\text{eff}})^{-N}}{1-(1+r_{\text{eff}})^{-1}}$$

or

$$A_P, 2 = A_P + (P_1 - P) \cdot \frac{(1+r_{\text{eff}})^N - 1}{r_{\text{eff}}}$$

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