Electromagnetic Field Theory - A Modern Tensorial Approach

Adrianus T. de Hoop #1

Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science,
Laboratory of Electromagnetic Research, Mekelweg 4, 2628 CD Delft, the Netherlands
[1] a.t.dehoop@tudelft.nl

***Abstract*—A modern tensorial approach to electromagnetic field theory in affine $(3+1)$-spacetime $\mathbb{R}^3 \times \mathbb{R}$ is presented. In it, the electric field and source quantities occur as tensors of rank one (vectors) and the magnetic field and source quantities as antisymmetric tensors of rank two. The approach leads to considerable simplifications over the traditional electric/magnetic vector approach.**

I. Introduction

A modern time-domain approach to electromagnetic (EM) field theory in affine $(3+1)$-spacetime is presented. In it, spacetime is conceived as an affine space $\mathbb{R}^3 \times \mathbb{R}$. No metric is assumed, like the Lorentzian one in traditional special relativity [1, Chapter IV]. In accordance with the Einstein axiom on the tensorial structure of macroscopic physical quantities in space, the electric field and source quantities occur as tensors of rank one (vectors) and the magnetic field and source quantities as antisymmetric tensors of rank two. In accordance with the Lorentz theory of electrons, the presence of matter is introduced via source quantities in an unbounded, homogeneous, isotropic background medium (vacuum) [2]. The role of the field/source compatibility relations, the constitutive relations, the EM energy balance and the EM balance of momentum are discussed, and expressions are derived for the field radiated by sources. In view of their numerous applications, also the field/source/contrast-in-medium reciprocity relations of the time-convolution type and the time-correlation type are given. The final section lists a number of conclusions, some of them with far-reaching consequences. One of them is that, in the approach, only elementary mathematics and some array theory is needed and standard vector calculus is superfluous, while the much discussed concept of 'magnetic monopole' is shown to need a reformulation.

II. The EM source quantities and the EM field equations

Standard macroscopic wavefield theory is characterized by the occurrence of two *intensive field quantities* whose 'inner product' represents the transfer of field energy and two *extensive field quantities* whose 'inner product' represents the transfer of field momentum, together with the occurrence of two field-independent *source quantities* that are representative of the excitation of the field. According to [3], all these quantities are spatial tensors in $\mathbb{R}^3$ and can depend on time in $\mathbb{R}$. In the field equations, the spatial changes of the intensive field quantities are counterbalanced by a change in time of the extensive field quantities, with the consequence that *wavelike solutions* can occur.

To specify an observer's position in space, we employ the Cartesian coordinates $\{x_1, x_2, x_3\}$ with respect to a reference frame with the origin $\mathcal{O}$ and the mutually perpendicular base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$. The corresponding position vector is $\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3$. The time coordinate is t. The subscript notation for vectors and tensors is employed and the summation convention applies to repeated subscripts in a single term in an expression. Lower-case Latin symbols are used for this purpose; they run through the values $\{1,2,3\}$. Partial differentiation with respect to x_m is denoted by ∂_m; ∂_t is a reserved symbol for differentiation with respect to t.

The *electric field strength* $E_r(\boldsymbol{x},t)$, a tensor of rank one, and the *magnetic field strength* $H^-_{p,q}(\boldsymbol{x},t) = -H^-_{q,p}(\boldsymbol{x},t)$, an antisymmetric tensor of rank two, are the intensive field quantities in EM theory; their 'inner product'

$$S_p = H^-_{p,q}E_q \tag{1}$$

represents the *area density of EM power flow* (Poynting vector) [4, Section 21.2], [5, Section 2.19]. The *electric flux density* $D_k(\boldsymbol{x},t)$, a tensor of rank one, and the *magnetic flux density* $B^-_{i,j}(\boldsymbol{x},t) = -B^-_{j,i}(\boldsymbol{x},t)$, an antisymmetric tensor of rank two, are the extensive field quantities in EM theory; their 'inner product'

$$G_i = B^-_{i,j}D_j \tag{2}$$

represents the *volume density of EM momentum* [5, Section 2.6].

The EM field equations that couple the spatial and temporal changes are

$$\partial_m H^-_{m,k} + \partial_t D_k = J_k, \tag{3}$$

$$[\partial_i E_j]^- + \partial_t B^-_{i,j} = K^-_{i,j}, \tag{4}$$

where J_k is the *volume density of impressed electric current*, $K^-_{i,j}$ is the *volume density of impressed magnetic current*, and

$$[\partial_i E_j]^- = (\partial_i E_j - \partial_j E_i)/2. \tag{5}$$

978-1-4244-5154-8/10/$26.00 ©2010 IEEE

III. The field/source compatibility relations, the volume densities of charge, the uniqueness of the initial-value problem and the constitutive relations

Operating on (3) with ∂_k and taking into account that $\partial_k \partial_m H^-_{m,k} = 0$ yields the *field/source compatibility relation*

$$\partial_t(\partial_k D_k) = \partial_k J_k. \tag{6}$$

Operating on (4) with ∂_k and observing that $\partial_k[\partial_i E_j]^- + \partial_i[\partial_j E_k]^- + \partial_j[\partial_k E_i]^- = 0$, yields the *field/source compatibility relation*

$$\begin{aligned}&\partial_t(\partial_k B^-_{i,j} + \partial_i B^-_{j,k} + \partial_j B^-_{k,i}) = \\ &\partial_k K^-_{i,j} + \partial_i K^-_{j,k} + \partial_j K^-_{k,i} \\ &\text{for } i \neq j \neq k.\end{aligned} \tag{7}$$

To accommodate the interpretation of the volume densities of current to consist of collections of moving charged particles, we introduce as the *volume density of electric charge* the scalar quantity

$$\rho = -\partial_t^{-1}(\partial_k J_k) \tag{8}$$

and as the *volume density of magnetic charge* the completely anti-symmetric tensor of rank three

$$\begin{aligned}\sigma_{i,j,k} &= -\partial_t^{-1}(\partial_k K^-_{i,j} + \partial_i K^-_{j,k} + \partial_j K^-_{k,i}) \\ &\quad \text{for } i \neq j \neq k,\end{aligned} \tag{9}$$

where ∂_t^{-1} denotes integration with respect to time.

In (3) – (4), the number of unknowns is twice the number of equations. To meet the physical requirement of the *uniqueness* of the EM *initial-value problem* [6], the field equations are supplemented with the *constitutive relations*

$$D_k = \epsilon\, E_k, \tag{10}$$

$$B^-_{i,j} = \mu\, H^-_{i,j}, \tag{11}$$

in which ϵ and μ are taken to be positive constants to fit within the Lorentz theory of electrons [2], in which theory J_k and $K^-_{i,j}$ represent the presence of matter in an unbounded homogeneous, isotropic, empty universe. (Note that also the special theory of relativity is based on this concept [1, Chapter V].)

IV. The EM energy balance

Replacing D_k in (3) with ϵE_k and operating on the resulting equation with E_k, replacing $B^-_{i,j}$ in (4) with $\mu H^-_{i,j}$ and operating on the resulting equation with $H^-_{i,j}$, and adding the results, we arrive at the local form of the *EM energy balance* that interrelates the *area density of EM power flow* S_m, the *volume density of stored EM energy* w^{EH} and the *volume density of EM power delivered by the sources* P^{JK} via

$$\partial_m S_m + \partial_t w^{EH} = P^{JK}, \tag{12}$$

with

$$S_m = H^-_{m,k} E_k, \tag{13}$$

$$w^{EH} = \epsilon E_k E_k/2 + \mu H^-_{i,j} H^-_{i,j}/2, \tag{14}$$

$$P^{JK} = J_k E_k + K^-_{i,j} H^-_{i,j}. \tag{15}$$

The global form of (12) for a bounded domain $\mathcal{D}$, with boundary $\partial\mathcal{D}$, follows upon integrating (12) over $\mathcal{D}$ and applying Gauss' theorem [4, Chapter 21].

V. The EM balance of momentum

Replacing $H^-_{i,j}$ in (3) with $\mu^{-1} B^-_{i,j}$ and operating on the resulting equation with $B^-_{j,k}$, replacing E_i in (4) with $\epsilon^{-1} D_i$ and operating on the resulting equation with D_j, and adding the results, we arrive, after a number of rearrangements of the different terms and the use of the compatibility relations (6) and (7), at the local form of the *EM balance of momentum* that interrelates the *Maxwell stress tensor* $T^{DB}_{i,j}$, the *volume density of EM momentum* G^{DB}_j and the *volume density of EM force* f^{JK}_j via

$$-\partial_i T^{DB}_{i,j} + \partial_t G^{DB}_j = f^{JK}_j, \tag{16}$$

with

$$\begin{aligned}T^{DB}_{i,j} &= [D_i D_j - (D_k D_k/2)\delta_{i,j}]/\epsilon + \\ &\quad [-B^-_{i,k} B^-_{j,k} + (B^-_{m,k} B^-_{m,k}/4)\delta_{i,j}]/\mu,\end{aligned} \tag{17}$$

$$G^{DB}_i = B^-_{i,j} D_j, \tag{18}$$

$$\begin{aligned}f^{JK}_j &= B^-_{j,k} J_k + K^-_{j,k} D_k - D_j \rho/\epsilon - \\ &\quad B^-_{m,k} \sigma_{m,k,j}/2\mu,\end{aligned} \tag{19}$$

where (8) and (9) have been used. The global form of (16) for a bounded domain $\mathcal{D}$, with boundary $\partial\mathcal{D}$, follows upon integrating (16) over $\mathcal{D}$ and applying Gauss' theorem.

VI. EM radiation from sources

Eliminating D_k, $B^-_{i,j}$ and $H^-_{i,j}$ from (3) – (4) and (10) – (11), using the field/source compatibility relation (6), we arrive at the *electric-field vector wave equation*

$$\begin{aligned}(\partial_m \partial_m) E_k - c^{-2} \partial_t^2 E_k &= -F_k \\ &= -\delta(\boldsymbol{x}, t) \overset{(\boldsymbol{x})}{*} \overset{(t)}{*} F_k,\end{aligned} \tag{20}$$

with

$$c = (2\epsilon\mu)^{-1/2}, \tag{21}$$

$$F_k = 2\mu \partial_t J_k - \epsilon^{-1} \partial_t^{-1} \partial_k (\partial_m J_m) - 2\partial_m K^-_{m,k}, \tag{22}$$

and where $\overset{(\boldsymbol{x})}{*}$ denotes spatial convolution over $\mathbb{R}^3$ and $\overset{(t)}{*}$ temporal convolution over $\mathbb{R}$. The solution of (20) can be written as

$$E_k = G \overset{(\boldsymbol{x})}{*} \overset{(t)}{*} F_k, \tag{23}$$

where

$$(\partial_m \partial_m) G - c^{-2} \partial_t^2 G = -\delta(\boldsymbol{x}, t), \tag{24}$$

with the solution

$$G(\boldsymbol{x},t) = \delta(t-|\boldsymbol{x}|/c)/4\pi|\boldsymbol{x}| \tag{25}$$

as the 3D Green's function of the scalar wave equation. Re-using (4) in which $B^-_{i,j}$ has been replaced with $\mu H^-_{i,j}$ yields the corresponding expression for the magnetic field.

Introducing the *electric-current vector potential*

$$A_k = G \overset{(\boldsymbol{x})(t)}{*\;*} J_k \tag{26}$$

and the *magnetic-current tensor potential*

$$\Psi^-_{i,j} = G \overset{(\boldsymbol{x})(t)}{*\;*} K^-_{i,j}, \tag{27}$$

we arrive at

$$E_k = 2\mu\partial_t A_k - \epsilon^{-1}\partial_t^{-1}\partial_k\partial_m A_m - 2\partial_m\Psi^-_{m,k} \tag{28}$$

and

$$\begin{aligned} H^-_{i,j} &= \mu^{-1}\partial_t^{-1}K^-_{i,j} - 2[\partial_i A_j]^- + \\ &\quad 2\mu^{-1}\partial_t^{-1}[\partial_i\partial_m\Psi^-_{m,j}]^- \qquad (29)\\ &= -2[\partial_i A_j]^- + 2\epsilon\partial_t\Psi^-_{i,j} - \\ &\quad \mu^{-1}\partial_t^{-1}\partial_m[\partial_m\Psi^-_{i,j} + \partial_i\Psi^-_{j,m} + \partial_j\Psi^-_{m,i}], \qquad (30) \end{aligned}$$

where the tensor wave equation for the magnetic tensor potential has been used.

A. Far-field approximation

For radiating sources of bounded spatial support, the *far-field approximation* to (26) – (30) follows upon using in the spatial convolution integrals [4, Sections 26.11, 26.12]

$$\begin{aligned} |R\boldsymbol{\xi} - \boldsymbol{x}'| &= R - \xi_m x'_m + O(R^{-1}) \text{ as } R\to\infty,\\ &\quad \text{with } \xi_m = x_m/|\boldsymbol{x}| \text{ and}\\ &\quad \boldsymbol{x}' \in \text{supp}(J_k, K^-_{i,j}), \end{aligned} \tag{31}$$

together with $\partial_m \simeq -(\xi_m/c)\partial_t$, and the representations

$$\begin{aligned} &\{E_k, H^-_{i,j}, A_k, \Psi^-_{i,j}\}(\boldsymbol{x},t) = \\ &\frac{\{E^\infty_k, H^{-;\infty}_{i,j}, A^\infty_k, \Psi^{-;\infty}_{i,j}\}(\boldsymbol{\xi}, t-R/c)}{4\pi R}\,[1+O(R^{-1}]\\ &\text{as } R\to\infty, \end{aligned} \tag{32}$$

with

$$A^\infty_k(\boldsymbol{\xi},t) = \int_{\boldsymbol{x}'\in\text{supp}(J_k)} J_k(\boldsymbol{x}', t+\xi_m x'_m/c)\mathrm{d}V(\boldsymbol{x}'), \tag{33}$$

$$\Psi^{-;\infty}_{i,j}(\boldsymbol{\xi},t) = \int_{\boldsymbol{x}'\in\text{supp}(K^-_{i,j})} K^-_{i,j}(\boldsymbol{x}', t+\xi_m x'_m/c)\mathrm{d}V(\boldsymbol{x}'), \tag{34}$$

and

$$E^\infty_k = 2\mu\partial_t(\delta_{k,m} - \xi_k\xi_m)A^\infty_m + 2(\xi_m/c)\partial_t\Psi^{-;\infty}_{m,k}, \tag{35}$$

$$\begin{aligned} H^\infty_{i,j} &= (\xi_i/c)\partial_t A^\infty_j - (\xi_j/c)\partial_t A^\infty_i + \\ &\quad 2\epsilon[(\xi_i/c)(\xi_m/c)\partial_t\Psi^{-;\infty}_{m,j} - (\xi_j/c)(\xi_m/c)\partial_t\Psi^{-;\infty}_{m,i}]. \end{aligned} \tag{36}$$

The far-field amplitudes satisfy the relations $(\xi_m/c)H^{-;\infty}_{m,k} = \epsilon E^\infty_k$ and $[\xi_i E^\infty_k]^- = \mu H^{-;\infty}_{i,j}$, that are characteristic for the (local) behavior as a plane wave propagating in the direction of ξ_m.

VII. Field/source/contrast-in-medium reciprocity relation of the time-convolution type

A reciprocity theorem interrelates the fields, source distributions and constitutive properties of two 'states' that can occur in one and the same domain in space. The two states will be referred to as 'State A' and 'State B', respectively. Two types of reciprocity relations appear to be useful: one of the time-convolution type, one of the time-correlation type [4, Chapter 28]. In view of this, we explicitly indicate the time argument in the quantities involved.

For the reciprocity relation of the time-convolution type, State A is characterized by the EM field equations

$$\partial_m H^{-;A}_{m,k}(t) + \epsilon^A\partial_t E^A_k(t) = J^A_k(t), \tag{37}$$

$$[\partial_i E^A_j(t)]^- + \mu^A\partial_t H^{-;A}_{i,j}(t) = K^{-;A}_{i,j}(t), \tag{38}$$

and State B by the field equations

$$\partial_m H^{-;B}_{m,k}(t) + \epsilon^B\partial_t E^B_k(t) = J^B_k(t), \tag{39}$$

$$[\partial_i E^B_j(t)]^- + \mu^B\partial_t H^{-;B}_{i,j}(t) = K^{-;B}_{i,j}(t). \tag{40}$$

Operating on (37) with $\overset{(t)}{*} E^B_k$, on (38) with $-\overset{(t)}{*} H^{-;B}_{i,j}$, on (39) with $-\overset{(t)}{*} E^A_k$ and on (40) with $\overset{(t)}{*} H^{-;A}_{i,j}$, and adding the results. we arrive at the *local form of the field/source/contrast-in-medium reciprocity relation of the time-convolution type*

$$\begin{aligned} &\partial_m[H^{-;A}_{m,k}(t) \overset{(t)}{*} E^B_k(t) - H^{-;B}_{m,k}(t) \overset{(t)}{*} E^A_k(t)] + \\ &\partial_t(\epsilon^A - \epsilon^B)E^A_k(t) \overset{(t)}{*} E^B_k(t) + \\ &\partial_t(\mu^B - \mu^A)H^{-;B}_{i,j}(t) \overset{(t)}{*} H^{-;A}_{i,j}(t) = \\ &J^A_k(t) \overset{(t)}{*} E^B_k(t) + K^{-;B}_{i,j}(t) \overset{(t)}{*} H^{-;A}_{i,j}(t) - \\ &J^B_k(t) \overset{(t)}{*} E^A_k(t) - K^{-;A}_{i,j}(t) \overset{(t)}{*} H^{-;B}_{i,j}(t). \end{aligned} \tag{41}$$

The *global form* of the reciprocity relation, applying to the bounded domain $\mathcal{D}$, follows upon integrating (41) over the domain $\mathcal{D}$ and applying Gauss' theorem (Figure 1). This leads to

$$\begin{aligned} &\int_{\partial\mathcal{D}} \nu_m[H^{-;A}_{m,k}(t) \overset{(t)}{*} E^B_k(t) - H^{-;B}_{m,k}(t) \overset{(t)}{*} E^A_k(t)]\mathrm{d}A + \\ &\int_{\mathcal{D}}[\partial_t(\epsilon^A - \epsilon^B)E^A_k(t) \overset{(t)}{*} E^B_k(t) + \\ &\partial_t(\mu^B - \mu^A)H^{-;B}_{i,j}(t) \overset{(t)}{*} H^{-;A}_{i,j}(t)]\mathrm{d}V = \\ &\int_{\mathcal{D}}[J^A_k(t) \overset{(t)}{*} E^B_k(t) + K^{-;B}_{i,j}(t) \overset{(t)}{*} H^{-;A}_{i,j}(t) - \\ &J^B_k(t) \overset{(t)}{*} E^A_k(t) - K^{-;A}_{i,j}(t) \overset{(t)}{*} H^{-;B}_{i,j}(t)]\mathrm{d}V, \end{aligned} \tag{42}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$ and ν_m is the unit vector along the outward normal to $\partial\mathcal{D}$. This relation is a basic tool in the theory of EM imaging and inversion methods [7].

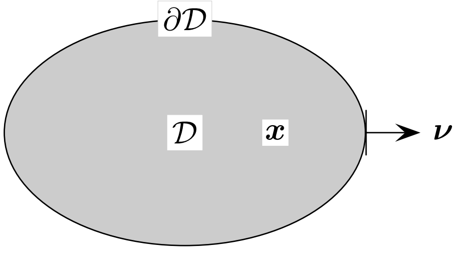

Fig. 1. Domain of application of global reciprocity.

VIII. Field/source/contrast-in-medium reciprocity relation of the time-correlation type

For the reciprocity relation of the time-correlation type, State A is characterized by the EM field equations

$$\partial_m H^{-;A}_{m,k}(t) + \epsilon^A \partial_t E^A_k(t) = J^A_k(t), \tag{43}$$

$$[\partial_i E^A_j(t)]^- + \mu^A \partial_t H^{-;A}_{i,j}(t) = K^{-;A}_{i,j}(t), \tag{44}$$

State B by the field equations

$$\partial_m H^{-;B}_{m,k}(-t) - \epsilon^B \partial_t E^B_k(-t) = J^B_k(-t), \tag{45}$$

$$[\partial_i E^B_j(-t)]^- - \mu^B \partial_t H^{-;B}_{i,j}(-t) = K^{-;B}_{i,j}(-t). \tag{46}$$

Operating on (43) with $\overset{(t)}{*}\; E^B_k(-t)$, on (44) with $\overset{(t)}{*}\; H^{-;B}_{i,j}(-t)$, on (45) with $\overset{(t)}{*}\; E^A_k(t)$ and on (46) with $\overset{(t)}{*}\; H^{-;A}_{i,j}(t)$, and adding the results. we arrive at the *local form of the field/source/contrast-in-medium reciprocity relation of the time-correlation type*

$$\begin{aligned}
&\partial_m[H^{-;A}_{m,k}(t) \overset{(t)}{*} E^B_k(-t) + H^{-;B}_{m,k}(-t) \overset{(t)}{*} E^A_k(t)] + \\
&\partial_t(\epsilon^A - \epsilon^B) E^A_k(t) \overset{(t)}{*} E^B_k(-t) + \\
&\partial_t(\mu^A - \mu^B) H^{-;B}_{i,j}(-t) \overset{(t)}{*} H^{-;A}_{i,j}(t) = \\
&J^A_k(t) \overset{(t)}{*} E^B_k(-t) + K^{-;B}_{i,j}(-t) \overset{(t)}{*} H^{-;A}_{i,j}(t) + \\
&J^B_k(-t) \overset{(t)}{*} E^A_k(t) + K^{-;A}_{i,j}(t) \overset{(t)}{*} H^{-;B}_{i,j}(-t).
\end{aligned} \tag{47}$$

The *global form* of the reciprocity relation, applying to the bounded domain $\mathcal{D}$, follows upon integrating (47) over the domain $\mathcal{D}$ and applying Gauss' theorem (Figure 1). This leads to

$$\begin{aligned}
&\int_{\partial\mathcal{D}} \nu_m [H^{-;A}_{m,k}(t) \overset{(t)}{*} E^B_k(-t) + H^{-;B}_{m,k}(-t) \overset{(t)}{*} E^A_k(t)] \mathrm{d}A + \\
&\int_{\mathcal{D}} [\partial_t(\epsilon^A - \epsilon^B) E^A_k(t) \overset{(t)}{*} E^B_k(-t) + \\
&\partial_t(\mu^A - \mu^B) H^{-;B}_{i,j}(-t) \overset{(t)}{*} H^{-;A}_{i,j}(t) \mathrm{d}V = \\
&\int_{\mathcal{D}} [J^A_k(t) \overset{(t)}{*} E^B_k(-t) + K^{-;B}_{i,j}(-t) \overset{(t)}{*} H^{-;A}_{i,j}(t) + \\
&J^B_k(-t) \overset{(t)}{*} E^A_k(t) + K^{-;A}_{i,j}(t) \overset{(t)}{*} H^{-;B}_{i,j}(-t)] \mathrm{d}V.
\end{aligned} \tag{48}$$

This relation is a basic one in the theory of Green's function retrieval methods through time cross-correlation [8].

IX. Conclusion

The tensorial approach of EM field theory offers considerable simplifications over the standard approach via the electric/magnetic vector one. It involves only elementary mathematics, makes standard vector calculus superfluous, and needs no specification as regards the orientation of the spatial reference frame, while computationally it maps straightforwardly on 'array' handling techniques in electromagnetic software. An important issue of a physical nature is that it puts the discussion on the existence of a 'magnetic monopole' on a different level. The theory naturally leads to the concept of 'magnetic charge' as a completely antisymmetric tensor of rank three and not to one of a scalar quantity. In view of this, experiments to detect the charge through its 'sensing' by an EM field have to be reconsidered within the full tensorial framework. An even more fundamental deviation from the standard theory (in particular as far as the special theory of relativity is concerned) is that spacetime is conceived as an affine space rather than a metric space, so, also the four-dimensional Lorentz metric goes out of sight, together with its geometric consequences. The view of affinity is also supported by the fact that the principle of causality applies to the (macroscopic) physical (EM) field (uniqueness of the initial-value problem), which is a principle that only applies to the time coordinate and not to the spatial coordinates.

Finally, it is observed that the magnetic constitutive coefficient introduced in (11) differs by a factor of two from the traditional one in magnetic vector theory. It is, however, logically consistent with the present tensorial theory.

References

[1] C. Moeller, *The Theory of Relativity*. Oxford: Clarendon Press, 1955.

[2] H. A. Lorentz, *The Theory of Electrons and its Applications to the Phenomena of Light and Radiant Heat*. Leipzig: Teubner, 1916.

[3] A. Einstein, *The Meaning of Relativity*. Princeton NJ: Princeton University Press, 1955.

[4] A. T. de Hoop, *Handbook of Radiation and Scattering of Waves*. London: Academic Press, 1995, electronic reproduction with corrections, 2008, freely downlaodable, for private use, from www.atdehoop.com.

[5] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw–Hill, 1941.

[6] A. T. de Hoop, "A time-domain uniqueness theorem for electromagnetic wavefield modeling in dispersive, anisotropic media," *The Radio Science Bulletin, URSI*, no. 305, pp. 17–21, 2003.

[7] P. M. van den Berg, H. Blok, and J. T. Fokkema, Eds., *Wavefields and Reciprocity*. Delft, the Netherlands: Delft University Press, 1996, proceedings of a Symposium held in honour of Professor dr. A. T. de Hoop, November 20–21, 1996, Delft, the Netherlands, 150 pp.

[8] R. Snieder, K. Wapenaar, and U. Wegler, "Unified green's function retrieval by cross-correlation; connection with energy principles," *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, vol. 75, p. 036103, 2007.